

Arithmetic Groups, Base Change, and Representation Growth

Nir Avni, Benjamin Klopsch, Uri Onn, Christopher Voll

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Abstract

Let \mathbf{G} be a semi-simple algebraic group defined over the ring of integers O_K in a number field K , and let $L \supset K$ be a finite extension with ring of integers O_L . We relate the asymptotic behavior of the number of irreducible complex representations of $\mathbf{G}(O_K)$ of dimension less than n , as n tends to infinity, with the corresponding asymptotic for $\mathbf{G}(O_L)$, assuming both groups satisfy the weak Congruence Subgroup Property. More precisely, we show that both sequences grow polynomially with the same exponent. As a corollary, we prove a conjecture of Larsen and Lubotzky regarding the representation growth of lattices in higher rank semisimple groups.

1 Introduction

1.1 The main results

The goal of this paper is to prove, in characteristic 0, a conjecture of Michael Larsen and Alex Lubotzky concerning the representation growths of irreducible lattices in higher rank semi-simple groups. By the representation growth of a group G we mean the asymptotic behavior of the sequence $R_n(G)$, $n \in \mathbb{N}$, where $R_n(G)$ is the number of equivalence classes of irreducible complex representations of G whose dimension is less than or equal to n . According to Margulis' Arithmeticity Theorem, the lattices in question are commensurable to groups of the form $\mathbf{G}(O_S)$. Here O is the ring of integers in a number field K , S is a finite set of valuations of K , O_S is the ring of S -integers, \mathbf{G} is an affine group scheme over O_S whose generic fiber is connected, simply-connected absolutely almost simple, and we assume that $\text{rank}_S(\mathbf{G}) = \sum_{v \in S} \text{rank}_{K_v}(\mathbf{G}) \geq 2$ and that the infinite places that are in S are precisely those for which $\text{rank}_{K_v}(\mathbf{G}) \geq 1$. Recall that, by a theorem of Borel

and Harish-Chandra, the image of $\mathbf{G}(O_S)$ under the diagonal embedding into the semi-simple group $G = \prod_{v \in S} \mathbf{G}(K_v)$ is indeed a lattice. For precise notions and more complete description, see [28]. For short, we call a group arithmetic if it is commensurable to a group of the form $\mathbf{G}(O_S)$ as above. We stress that our arithmetic groups are always in characteristic 0.

The study of representation growth of arithmetic groups started with the paper [25] of Lubotzky and Martin. They show that, for an arithmetic group Δ , the sequence of numbers $R_n(\Delta)$ is bounded by some polynomial in n if and only if Δ has the (weak) Congruence Subgroup Property. To discuss the latter, we denote the completion of a ring O in a prime ideal \mathfrak{p} by $O_{\mathfrak{p}}$. The group $\mathbf{G}(O_S)$ has the weak Congruence Subgroup Property if the kernel of the natural map $\widehat{\mathbf{G}(O_S)} \rightarrow \mathbf{G}(\widehat{O_S})$, from the pro-finite completion, $\widehat{\mathbf{G}(O_S)}$, to the pro-congruence completion, $\mathbf{G}(\widehat{O_S}) = \prod_{\mathfrak{p} \in \text{Spec}(O) \setminus S} \mathbf{G}(O_{\mathfrak{p}})$, is finite. In particular, this theorem of Lubotzky and Martin allows one to regard the weak Congruence Subgroup Property as a group theoretic property, and not merely an arithmetic one. From now on, we restrict our attention to arithmetic lattices that satisfy the weak Congruence Subgroup Property; by a conjecture of Serre (which is known in most cases), this holds whenever the S -rank of \mathbf{G} , $\text{rank}_S(\mathbf{G})$, is greater than one, and, for all finite $\mathfrak{p} \in S$, the group $\mathbf{G}(K_{\mathfrak{p}})$ is not compact. In particular, the property of having the weak Congruence Subgroup Property depends on the ambient group $\prod_{v \in S} \mathbf{G}(K_v)$ and not on the lattice itself. For more information on the Congruence Subgroup Property and Serre's conjecture, see [28, Chapter 9.5], [29] and the references therein.

Our aim in this paper is to prove quantitative results regarding representation growths of arithmetic groups in characteristic 0 having the weak Congruence Subgroup Property. In [3] it is shown that, if Δ is such a group, then there is a rational number $\alpha(\Delta)$ such that $R_n(\Delta) = n^{\alpha(\Delta) + o(1)}$, as n tends to infinity. We call $\alpha(\Delta)$ the degree of representation growth. It is closely related to the representation zeta function of Δ , which we define next.

Definition 1.1. *Suppose that G is a group such that $R_n(G)$ is finite for any positive integer n . The representation zeta function of G is the Dirichlet generating series*

$$\zeta_G(s) = \sum_{\chi \in \text{Irr}(G)} (\dim \chi)^{-s},$$

where $\text{Irr}(G)$ is the set of isomorphism classes of finite-dimensional irreducible complex representations of G and $s \in \mathbb{C}$.

For an arithmetic group Δ having the weak Congruence Subgroup Property, the theorem of Lubotzky and Martin mentioned above implies that $\zeta_{\Delta}(s)$ converges absolutely

in some non-empty right half-plane. The infimum of the set of real numbers σ such that $\zeta_\Delta(s)$ converges for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq \sigma$ is called the abscissa of convergence of $\zeta_\Delta(s)$, or of Δ , and is equal to $\alpha(\Delta)$; if the series $\zeta_G(s)$ diverges for all $s \in \mathbb{C}$, we write $\alpha(G) = \infty$. In general, we denote the abscissa of convergence of $\zeta_G(s)$ by $\alpha(G)$. It is known that $\alpha(G)$ is an invariant of the commensurability class of G ; we prove a more general claim in Lemma 3.2. Only very few explicit values of abscissae of convergence of arithmetic groups are known. If \mathbf{G} is a form of a power of SL_2 , the abscissa is 2, see [23, Theorem 10.1]. If \mathbf{G} is a form of a power of SL_3 , the abscissa is 1, see [4, Theorem C].

If Δ is an arithmetic lattice, then, following Lubotzky and Martin, one may regard $\alpha(\Delta)$ as a quantitative measure of the weak Congruence Subgroup Property. In [23, Conjecture 1.5], the authors suggest that it too depends only on the ambient group. In this paper, we prove this conjecture in characteristic 0.

Theorem 1.2. *Let H be a higher-rank semi-simple group in characteristic 0 (i.e. $H = \prod_{i=1}^n \mathbf{G}_i(K_i)$, where each K_i is a local field, each \mathbf{G}_i is an absolutely almost simple K_i -group, and $\sum \operatorname{rank}_{K_i} \mathbf{G}_i \geq 2$). If $\Delta_1, \Delta_2 \subset H$ are irreducible lattices and $\alpha(\Delta_1), \alpha(\Delta_2) < \infty$, then $\alpha(\Delta_1) = \alpha(\Delta_2)$.*

By the aforementioned theorem of Lubotzky and Martin, the assumption on the finiteness of the abscissae of convergence can be dropped if we assume Serre's conjecture. Using Margulis' Arithmeticity Theorem, Theorem 1.2 follows from the following:

Theorem 1.3. *Let Δ_1 and Δ_2 be two irreducible arithmetic lattices in the same semi-simple group in characteristic 0. Assume that both Δ_1 and Δ_2 have the weak Congruence Subgroup Property. Then $\alpha(\Delta_1) = \alpha(\Delta_2)$.*

We deduce Theorem 1.3 by studying the behavior of the abscissae of convergence of arithmetic lattices under base change. Our main results are summarized in Theorems 1.4 and 1.5 below. In the following, when we use the notation $R_n(G)$, ζ_G , and $\alpha(G)$ for a topological group G , we only consider continuous representations.

Theorem 1.4. *Let K be a number field with ring of integers O , let S be a finite set of valuations, and let \mathbf{G} be a group scheme defined over O_S . Assume that the generic fiber of \mathbf{G} is simply connected and absolutely almost simple, and that $\mathbf{G}(O_S)$ has the weak Congruence Subgroup Property. Then $\alpha(\mathbf{G}(O_S)) = \alpha(\mathbf{G}(\widehat{O}))$.*

Theorem 1.5. *Let $K \subset L$ be number fields with ring of integers $O_K \subset O_L$, and let \mathbf{G} be a semi-simple algebraic group scheme defined over O_K . Then $\alpha(\mathbf{G}(\widehat{O_K})) = \alpha(\mathbf{G}(\widehat{O_L}))$.*

The proofs of Theorems 1.4 and 1.5 are given in the next section; combining them, we see that the abscissa of convergence of $\mathbf{G}(O)$ is independent of the ring O . In fact, it depends only on the root system of \mathbf{G} .

Corollary 1.6. *For every irreducible root system Φ there is a constant α_Φ such that, for every number field K with ring of integers O , every finite set S of places of K , and every connected, simply connected absolutely almost simple algebraic group \mathbf{G} over K with absolute root system Φ the following holds: if $\mathbf{G}(O_S)$ has the weak Congruence Subgroup Property, then $\alpha(\mathbf{G}(O_S)) = \alpha_\Phi$.*

Proof. Let \mathbf{S} be a Chevalley group with root system Φ , i.e. a split connected, simply connected algebraic group over \mathbb{Q} with root system Φ . Consider an arithmetic group $\mathbf{G}(O_S)$ with the weak Congruence Subgroup Property, as in the corollary. There is an extension L of K such that \mathbf{G} and \mathbf{S} are isogenous over L . Denoting the ring of integers of L by O_L , Theorems 1.5 and 1.4 imply that

$$\alpha(\mathbf{G}(O_S)) = \alpha(\mathbf{G}(\widehat{O})) = \alpha(\mathbf{G}(\widehat{O_L})) = \alpha(\mathbf{S}(\widehat{O_L})) = \alpha(\mathbf{S}(\widehat{\mathbb{Z}}))$$

depends only on Φ . □

We now show how Theorem 1.3 follows from Theorems 1.4 and 1.5.

Proof of Theorem 1.3. Let H be a semi-simple group in characteristic 0. By this we mean that $H = \prod_{j=1}^r \mathbf{H}_j(F_j)$, where each F_j is a local field of characteristic 0, and each \mathbf{H}_j is a connected, absolutely almost simple group defined over F_j . We assume further that no $\mathbf{H}_j(F_j)$ is compact and that Δ is an irreducible lattice in H .

By definition of arithmetic group, there is a number field K with ring of integers O , a finite set S of places of K , and a connected, simply connected absolutely almost simple algebraic group \mathbf{G} defined over K such that there is a continuous homomorphism $\psi : \prod_{v \in S} \mathbf{G}(K_v) \rightarrow H$ whose kernel and cokernel are compact, and such that $\psi(\mathbf{G}(O_S))$ is commensurable to Δ . Since $\mathbf{G}(O_S)$ is finitely generated, there is a finite-index subgroup of it that is isomorphic to a finite-index subgroup of Δ . Hence $\alpha(\Delta) = \alpha(\mathbf{G}(O_S))$. Fix a split \mathbb{Q} -form \mathbf{S} of the connected, simply connected simple algebraic group associated to \mathbf{H}_1 . The map ψ extends to a homomorphism between $\prod_{v \in S} \mathbf{G}(\mathbb{C})$ and $\prod_{j=1}^r \mathbf{H}_j(\mathbb{C})$. Since $\mathbf{G}(\mathbb{C})$ is almost simple, we get that \mathbf{G} and the groups \mathbf{H}_j are all isogenous to \mathbf{S} over \mathbb{C} , and, therefore over some finite extension of K .

Suppose that L is a finite extension over which there is an isogeny between \mathbf{G} and \mathbf{S} . Then,

$$\alpha(\Delta) = \alpha(\mathbf{G}(O_S)) = \alpha(\mathbf{G}(\widehat{O})) = \alpha(\mathbf{G}(\widehat{O_L})) = \alpha(\mathbf{S}(\widehat{O_L})) = \alpha(\mathbf{S}(\widehat{\mathbb{Z}})).$$

Thus, $\alpha(\Delta)$ is determined by \mathbf{S} , which in turn is determined by H . □

1.2 Notations

For the reader's convenience, we summarize our notation throughout the paper.

- G denotes a group, \mathbf{G} an affine group scheme. \mathfrak{g} denotes the Lie algebra of \mathbf{G} .
- Δ, Λ usually denote groups.
- K and L denote number fields with ring of integers O_K and O_L .
- \mathfrak{p} denotes a prime of O_K , \mathfrak{q} a prime of O_L .
- \mathcal{A} is a ring and $\xi_{a,q}$ is a function defined in Definition 2.6.
- F, k, Γ are the sorts of the Denef–Pas language of valued fields, see Section 3.
- \mathcal{X}, \mathcal{Y} are definable sets defined in Definitions 4.2 and 4.9.
- Π, Ξ are the relative orbit method functions. See the discussion after Definition 4.2.
- $\mathcal{L}, \mathcal{S}, \mathcal{R}, \tilde{\mathcal{L}}, \tilde{\mathcal{S}}$ are definable functions/families of Lie algebras/rings over \mathcal{X} and \mathcal{Y} . See the discussion after Theorem 4.3 and also Theorem 4.15
- $\text{Grass}(\mathfrak{g}), \text{Grass}(\mathfrak{g})^{\text{nilp}}$ are the Grassmannian of Lie subalgebras and of nilpotent Lie subalgebras. See Proposition 3.13

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2 Overview of The Proof of Theorem 1.4 and Theorem 1.5

The starting point of the proof of Theorem 1.4 is the Euler factorization of the representation zeta function of arithmetic lattices. Suppose, as in the introduction, that

- O is the ring of integers of a number field K , and S is a finite set of places of K .
- \mathbf{G} is an algebraic group scheme over the ring O_S of S -integers whose generic fiber is connected, simply connected and absolutely almost simple.

- $\Delta = \mathbf{G}(O_S)$ satisfies the weak Congruence Subgroup Property.

By [23, Proposition 4.6], there is a finite-index subgroup $\Lambda \subset \Delta$ such that

$$\zeta_\Lambda(s) = \zeta_{\mathbf{G}(\mathbb{C})}(s)^{[K:\mathbb{Q}]} \cdot \prod_{\mathfrak{p} \in \text{Spec}(O) \setminus S} \zeta_{\Lambda_{\mathfrak{p}}}(s), \quad (1)$$

where the product is over the primes of O_S and $\Lambda_{\mathfrak{p}}$ is the closure, in the \mathfrak{p} -adic topology, of the image of Λ under the embedding $\Delta \rightarrow \mathbf{G}(O_{\mathfrak{p}})$. Furthermore, the generating function $\zeta_{\mathbf{G}(\mathbb{C})}(s)$ counts only the rational representations, and the generating functions $\zeta_{\Lambda_{\mathfrak{p}}}(s)$ count only continuous representations. Since Λ has finite index in Δ , the Strong Approximation Theorem implies $\Lambda_{\mathfrak{p}}$ is open in $\mathbf{G}(O_{\mathfrak{p}})$, for every \mathfrak{p} , and $\Lambda_{\mathfrak{p}} = \mathbf{G}(O_{\mathfrak{p}})$, for all but finitely many primes.

We will use the following lemma repeatedly.

Lemma 2.1. *For any two groups G and H , $\zeta_{G \times H}(s) = \zeta_G(s) \cdot \zeta_H(s)$. In particular, $\alpha(G \times H) = \max\{\alpha(G), \alpha(H)\}$.*

Proof. The first claim follows from the fact that the irreducible representations of $G \times H$ are the tensor products of the irreducible representations of G and H . The second claim follows from the first. \square

It is well-known that the abscissa of convergence of groups is a commensurability invariant (see [25, Lemma 2.2]; we prove a more general result in Lemma 3.2). In particular, $\alpha(\Delta)$ is equal to $\alpha(\Lambda)$. By [23, Theorem 5.1] and [23, Proposition 6.6], we have $\alpha(\mathbf{G}(\mathbb{C})) \leq \alpha(\mathbf{G}(O_{\mathfrak{p}}))$, for any $\mathfrak{p} \in \text{Spec}(O_S)$. Therefore, (1) shows that $\alpha(\Delta)$ is equal to the abscissa of convergence of the product $\prod_{\mathfrak{p} \notin S} \zeta_{\Lambda_{\mathfrak{p}}}(s)$. By another application of the commensurability invariance, $\alpha(\Delta)$ is equal to the abscissa of convergence of $\prod_{\mathfrak{p} \notin S} \zeta_{\mathbf{G}(O_{\mathfrak{p}})}(s)$.

To finish the proof of Theorem 1.4 we need only to show that the abscissa of convergence of the product $\prod \zeta_{\mathbf{G}(O_{\mathfrak{p}})}(s)$, extending over all primes of O , is unchanged by omitting finitely many factors. This is a consequence of the following, more general, result.

Theorem 2.2. *Let K be a number field with ring of integers O_K , let \mathbf{G} be a semi-simple algebraic group scheme defined over O_K , and let L be a finite extension of K with ring of integers O_L . For every prime \mathfrak{q} of O_L , there are infinitely many primes \mathfrak{p} of O_K such that $\alpha(\mathbf{G}(O_{L,\mathfrak{q}})) \leq \alpha(\mathbf{G}(O_{K,\mathfrak{p}}))$.*

Theorem 2.2 is a consequence of [4, Theorem B], whose proof uses p -adic integrals. The connection to p -adic integrals is [4, Corollary 3.7] (see also [21, Lemma 4.1]) from which it follows that there is a natural number d , quantifier-free definable functions¹ ϕ_1, ϕ_2 such

¹The notion of quantifier-free definable function is explained in Section 3

that, for every finite extension L of K , every prime \mathfrak{q} of O_L , and every sufficiently large integer r , the representation zeta function of the r -th congruence subgroup $\mathbf{G}^{(r)}(O_{L,\mathfrak{q}})$ of the \mathfrak{q} -adic group $\mathbf{G}(O_{L,\mathfrak{q}})$ can be expressed as follows:

$$\zeta_{\mathbf{G}^{(r)}(O_{L,\mathfrak{q}})}(s) = |O_L/\mathfrak{q}|^{\dim \mathbf{G} \cdot r} \int_{O_{L,\mathfrak{q}}^d} |\phi_1(x)| \cdot |\phi_2(x)|^{-s} d\lambda(x), \quad (2)$$

where the absolute value in the integrand is the \mathfrak{q} -adic one, and λ is the Haar measure on the additive group $O_{L,\mathfrak{q}}^d$ with total measure one; we call such a measure normalized. Since the abscissae of convergence of $\mathbf{G}(O_{L,\mathfrak{q}})$ and its r -th congruence subgroup are equal, it is enough to replace the claim in Theorem 2.2 by a similar claim for integrals of the form (2). The main point is that the functions ϕ_i are independent of L and \mathfrak{q} , which allows us to compare the integrals for different fields and primes.

Remark 2.3. *Theorem 2.2 can be regarded as a local analog of Theorem 1.5. We explain why a more naive analog is false. Suppose that $K \subset L$ is a finite extension of number fields, and that \mathfrak{q} is a prime of O_L lying over a prime \mathfrak{p} of O_K . [4, Theorem B] implies that $\alpha(\mathbf{G}(O_{K,\mathfrak{p}})) \leq \alpha(\mathbf{G}(O_{L,\mathfrak{q}}))$, but, in contrast to the the global case, Theorem 1.5, the inequality can be strict. As an example of strict inequality, let D be a division algebra of degree d greater than 30 over a local field F , and let \mathbf{G} be the algebraic group of elements of norm 1 in D . By [23, Theorem 7.1], the abscissa of convergence of $\mathbf{G}(O_F)$ is $2/d$. However, if $F \subset E$ is an extension over which D splits, then $\alpha(\mathbf{G}(O_E)) \geq 1/15$ by [23, Theorem 8.1].*

We now move on to the proof of Theorem 1.5. The integral presentation of (2), read for $K = L$, is insufficient for analyzing the abscissa of convergence of an infinite product of the groups $\mathbf{G}(O_{\mathfrak{p}})$, because the product of the congruence subgroups is no longer of finite index in the product of the groups $\mathbf{G}(O_{\mathfrak{p}})$. Thus, in order to deal with infinitely many primes, we need to find expressions for the zeta functions of the groups $\mathbf{G}(O_{\mathfrak{p}})$ themselves. Unable to do so, we approximate those zeta functions in the following sense:

Definition 2.4. *Let $f = f(s) = \sum a_n n^{-s}$ and $g = g(s) = \sum b_n n^{-s}$ be Dirichlet generating series, i.e. Dirichlet series with integer coefficients $a_n, b_n \geq 0$. Let $C \in \mathbb{R}$. Suppose that $\sigma_0 \in \mathbb{R}_{\geq 0}$ is greater than or equal to the abscissae of convergence of f and g . We write*

$$f \lesssim_C g \quad \text{for } \sigma > \sigma_0$$

if, for every $\sigma \in \mathbb{R}$ with $\sigma > \sigma_0$, we have $f(\sigma) \leq C^{1+\sigma} g(\sigma)$. We write $f \lesssim_C g$, without specifying the domain, if f and g have the same abscissa of convergence α and if $f \lesssim_C g$ for $\sigma > \max\{0, \alpha\}$. We write $f \sim_C g$, if $f \lesssim_C g$ and $g \lesssim_C f$.

We will routinely use the easy fact that, if $f \lesssim_{C_1} g$ and $g \lesssim_{C_2} h$, then $f \lesssim_{C_1 C_2} h$.

Lemma 2.5. *Let f, g be Dirichlet generating series with abscissae of convergence α_f, α_g . Suppose that $f = \prod_{m=1}^{\infty} (1 + f_m)$ and $g = \prod_{m=1}^{\infty} (1 + g_m)$, where f_m, g_m are Dirichlet generating series with vanishing constant coefficients. Suppose further that there is $C \in \mathbb{R}$ such that $f_m \lesssim_C g_m$ for all m . Then $\alpha_f \leq \alpha_g$.*

Proof. The abscissa of convergence of a Dirichlet generating series is determined by its behavior on the real axis. Let $\sigma \in \mathbb{R}_{>0}$. Then $g(\sigma) = \prod_m (1 + g_m(\sigma))$ converges if and only if $\sum_m g_m(\sigma)$ converges. As $f_m \lesssim_C g_m$ for all m , the latter implies that $\sum_m f_m(\sigma)$ converges, which is equivalent to the convergence of $f(\sigma) = \prod_m (1 + f_m(\sigma))$. \square

We now introduce a semi-ring \mathcal{A} whose elements help to index certain approximations to Dirichlet generating functions.

Definition 2.6. 1. Let \mathcal{A} be the collection of finite subsets a of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$. We turn \mathcal{A} into a commutative semi-ring by defining addition to be union, and defining the product of $a, b \in \mathcal{A}$ to be $a \cdot b = \{v + u \mid v \in a, u \in b\}$.

2. For $a \in \mathcal{A}$ and $q \in \mathbb{N}_{>1}$, define a Dirichlet series $\xi_{a,q}(s) = \sum_{(n,m) \in a} q^{n-ms}$.

The following is a proto-type of our approximation theorem, Theorem 2.8, which is one of the ingredients in the proof of Theorem 1.5.

Theorem 2.7. *Let Φ be a reduced root system. There is a constant $C \in \mathbb{R}$ and an element $a \in \mathcal{A}$ such that, for any prime power q and for any connected semi-simple algebraic group \mathbf{G} over \mathbb{F}_q with root system Φ , we have*

$$\zeta_{\mathbf{G}(\mathbb{F}_q)} - |\mathbf{G}(\mathbb{F}_q)/[\mathbf{G}(\mathbb{F}_q), \mathbf{G}(\mathbb{F}_q)]| \sim_C \xi_{a,q}.$$

In particular, $\zeta_{\mathbf{G}(\mathbb{F}_q)} \sim_{C'} 1 + \xi_{a,q}$ for some constant C' .

Proof. Throughout the proof, all constants are going to depend only on Φ . The proof requires the Lusztig classification of irreducible representations of finite groups of Lie type. We will use the notation of [12, Chapter 13]. In particular, the dual group of \mathbf{G} is denoted by \mathbf{G}^* , and, for every semi-simple conjugacy class $(g) \subset \mathbf{G}^*(\mathbb{F}_q)$, the corresponding Lusztig series is denoted by $\mathcal{E}(\mathbf{G}(\mathbb{F}_q), (g)) \subset \text{Irr}(\mathbf{G}(\mathbb{F}_q))$. The elements of $\mathcal{E}(\mathbf{G}(\mathbb{F}_q), (1))$ are called the unipotent characters of $\mathbf{G}(\mathbb{F}_q)$, and we define the unipotent zeta function of $\mathbf{G}(\mathbb{F}_q)$ as

$$\zeta_{\mathbf{G}(\mathbb{F}_q)}^{\text{unip}}(s) = \sum_{\rho \in \mathcal{E}(\mathbf{G}(\mathbb{F}_q), (1))} (\dim \rho)^{-s}.$$

The irreducible characters of $\mathbf{G}(\mathbb{F}_q)$ decompose into the Lusztig series $\mathcal{E}(\mathbf{G}(\mathbb{F}_q), (g))$, where (g) ranges over the semi-simple conjugacy classes in $\mathbf{G}^*(\mathbb{F}_q)$. Therefore,

$$\zeta_{\mathbf{G}(\mathbb{F}_q)}(s) = \sum_{(g) \subset \mathbf{G}^*(\mathbb{F}_q)} \sum_{\rho \in \mathcal{E}(\mathbf{G}(\mathbb{F}_q), (g))} (\dim \rho)^{-s},$$

where the outer sum is over all semi-simple conjugacy classes in $\mathbf{G}^*(\mathbb{F}_q)$. We will treat the central conjugacy classes and the non-central conjugacy classes separately.

Note that the finite abelian group $Z(\mathbf{G}^*(\mathbb{F}_q))$ is dual to the abelianization of $\mathbf{G}(\mathbb{F}_q)$. If $g \in Z(\mathbf{G}^*(\mathbb{F}_q))$, then the elements of $\mathcal{E}(\mathbf{G}(\mathbb{F}_q), (g))$ are twists of the elements of $\mathcal{E}(\mathbf{G}(\mathbb{F}_q), (1))$ by the character of $\mathbf{G}(\mathbb{F}_q)$ corresponding to g . By [12, Proposition 13.20],

$$\sum_{g \in Z(\mathbf{G}^*(\mathbb{F}_q))} \sum_{\rho \in \mathcal{E}(\mathbf{G}(\mathbb{F}_q), (g))} (\dim \rho)^{-s} = |Z(\mathbf{G}^*(\mathbb{F}_q))| \zeta_{\mathbf{G}(\mathbb{F}_q)}^{unip}(s) = |Z(\mathbf{G}^*(\mathbb{F}_q))| \zeta_{\mathbf{G}^{ad}(\mathbb{F}_q)}^{unip}(s).$$

\mathbf{G}^{ad} is a direct product of simple groups, $\mathbf{S}_1, \dots, \mathbf{S}_m$. The unipotent representations of $\mathbf{G}^{ad}(\mathbb{F}_q)$ are the irreducible representations that appear in the alternating sum of the cohomologies of the Deligne–Lusztig variety of $\mathbf{G}^{ad}(\mathbb{F}_q)$. Since this last variety is equal to the product of the Deligne–Lusztig varieties of the groups $\mathbf{S}_i(\mathbb{F}_q)$, the Kunneth formula implies that $\zeta_{\mathbf{G}^{ad}(\mathbb{F}_q)}^{unip}(s) = \prod_i \zeta_{\mathbf{S}_i(\mathbb{F}_q)}^{unip}(s)$. By [5, Sections 13.8, 13.9], for any i , the dimensions of the unipotent representations of $\mathbf{S}_i(\mathbb{F}_q)$ are given by polynomials in q , depending only on Φ , and there is only one unipotent representation of constant dimension, namely, the trivial representation. It follows that there is $a \in \mathcal{A}$ and $D \in \mathbb{R}$ such that

$$\sum_{g \in Z(\mathbf{G}^*(\mathbb{F}_q))} \sum_{\rho \in \mathcal{E}(\mathbf{G}(\mathbb{F}_q), (g))} (\dim \rho)^{-s} - |Z(\mathbf{G}^*(\mathbb{F}_q))| \sim_D \xi_{a,q}(s). \quad (3)$$

Now consider the sum

$$\Sigma = \sum_{(g) \subset \mathbf{G}^*(\mathbb{F}_q) \setminus Z(\mathbf{G}^*(\mathbb{F}_q))} \sum_{\rho \in \mathcal{E}(\mathbf{G}(\mathbb{F}_q), (g))} (\dim \rho)^{-s}.$$

where the outer sum is over the non-central semi-simple conjugacy classes. Suppose that $g \in \mathbf{G}^*(\mathbb{F}_q)$ is a non-central semi-simple element. By [12, Theorem 13.23 and Remark 13.24], there is a bijection between $\mathcal{E}(\mathbf{G}(\mathbb{F}_q), (g))$ and $\mathcal{E}(C_{\mathbf{G}^*}(g)(\mathbb{F}_q), (1))$ such that, if $\rho \in \text{Irr}(\mathbf{G}(\mathbb{F}_q))$ is mapped to $\tau \in \text{Irr}(C_{\mathbf{G}^*}(g)(\mathbb{F}_q))$, then $\dim \rho = \frac{|\mathbf{G}(\mathbb{F}_q)|_{q'}}{|C_{\mathbf{G}^*}(g)(\mathbb{F}_q)|_{q'}} \dim \tau$, where $n_{q'}$ denotes the prime-to- q part of the number n . It follows that

$$\Sigma = \sum_{(g) \subset \mathbf{G}^*(\mathbb{F}_q) \setminus Z(\mathbf{G}^*(\mathbb{F}_q))} \left(\frac{|\mathbf{G}(\mathbb{F}_q)|_{q'}}{|C_{\mathbf{G}^*}(g)(\mathbb{F}_q)|_{q'}} \right)^{-s} \zeta_{C_{\mathbf{G}^*}(g)(\mathbb{F}_q)}^{unip}(s).$$

Suppose that $g \in \mathbf{G}^*(\mathbb{F}_q)$ is a non-central semi-simple element. Since $\zeta_{C_{\mathbf{G}^*}(g)(\mathbb{F}_q)}^{unip}(s)$ is the product of the unipotent zeta functions of the almost simple factors of $(C_{\mathbf{G}^*}(g))^{ad}(\mathbb{F}_q)$ (see the discussion before Equation (3)), and, by [24, Lemma 2.1], there is a constant C_1 , such that there are at most C_1 unipotent representations of $C_{\mathbf{G}^*}(g)(\mathbb{F}_q)$. Therefore, $\zeta_{C_{\mathbf{G}^*}(g)(\mathbb{F}_q)}^{unip}(s) \sim_{C_1} 1$. In addition, there is a constant C_2 such that, for any $g \in \mathbf{G}^*(\mathbb{F}_q)$, the number of connected components of $C_{\mathbf{G}^*}(g)$ is at most C_2 . It follows that

$$\left(\frac{|\mathbf{G}(\mathbb{F}_q)|_{q'}}{|C_{\mathbf{G}^*}(g)(\mathbb{F}_q)|_{q'}} \right)^{-s} \zeta_{C_{\mathbf{G}^*}(g)(\mathbb{F}_q)}^{unip}(s) \sim_{C_1 C_2} \left(\frac{|\mathbf{G}(\mathbb{F}_q)|_{q'}}{|C_{\mathbf{G}^*}(g)^0(\mathbb{F}_q)|_{q'}} \right)^{-s}$$

It is well known (see, for example [24, Lemma 2.2]) that $C_{\mathbf{G}^*}(g)$ is a reductive subgroup of \mathbf{G}^* and has maximal rank and that there are boundedly many conjugacy classes of connected reductive groups of maximal rank in \mathbf{G}^* . Fix representatives $\mathbf{H}_1, \dots, \mathbf{H}_N$ of the conjugacy classes of connected reductive groups of maximal rank in \mathbf{G}^* . Every semi-simple conjugacy class in $\mathbf{G}^*(\mathbb{F}_q)$ contains at least one element g such that $C_{\mathbf{G}^*}(g)^0 = \mathbf{H}_i$ for some i , and, by [24, Lemma 2.2], the conjugacy class contains at most C_3 such elements, for some constant C_3 . Therefore,

$$\Sigma \sim_{C_1 C_2 C_3} \sum_{i=1}^N |\{g \in \mathbf{G}^*(\mathbb{F}_q) \mid C_{\mathbf{G}^*}(g)^0 = \mathbf{H}_i\}| \left(\frac{|\mathbf{G}(\mathbb{F}_q)|_{q'}}{|\mathbf{H}_i(\mathbb{F}_q)|_{q'}} \right)^{-s}.$$

By [5, Proposition 3.5.1], if $g \in \mathbf{G}^*(\mathbb{F}_q)$ is semi-simple and $C_{\mathbf{G}^*}(g)^0 = \mathbf{H}_i$, then $g \in \mathbf{H}_i(\mathbb{F}_q)$. It follows that, for some constant C_4 , $|\{g \in \mathbf{G}^*(\mathbb{F}_q) \mid C_{\mathbf{G}^*}(g)^0 = \mathbf{H}_i\}| \sim_{C_4} q^{\dim Z(\mathbf{H}_i)}$. There is a constant C_5 such that $|\mathbf{G}(\mathbb{F}_q)|_{q'} \sim_{C_5} q^{\dim \mathbf{G} - |\Phi^+|}$ and, if the root system of \mathbf{H}_i is Ψ_i , then $|\mathbf{H}_i(\mathbb{F}_q)|_{q'} \sim_{C_5} q^{\dim \mathbf{H}_i - |\Psi_i^+|}$. Therefore,

$$\Sigma \sim_{C_1 C_2 C_3 C_4 C_5} \sum_{i=1}^N q^{\dim Z(\mathbf{H}_i) - (\dim \mathbf{G} + |\Psi_i^+| - |\Phi^+| - \dim \mathbf{H}_i)s}.$$

The root systems Ψ_i are exactly the sub root systems of Φ , and, if Ψ is a sub root system of Φ , then the number of subgroups \mathbf{H}_i with root system Ψ is bounded by some constant C_6 depending only on Φ (see [24, Lemma 2.2]). Therefore, if we denote $C' = C_1 C_2 C_3 C_4 C_5 C_6$, then there is an element $a \in \mathcal{A}$ such that $\Sigma \sim_{C'} \xi_{a,q}$. Combining this with (3), we get the first claim of the theorem.

The second claim follows from the first since the order of $\mathbf{G}(\mathbb{F}_q)/[\mathbf{G}(\mathbb{F}_q), \mathbf{G}(\mathbb{F}_q)]$, which is equal to the order of $Z(\mathbf{G}^*(\mathbb{F}_q))$, is bounded by some constant. \square

To formulate our next key result, recall that a subset T of primes of a ring of integers O in a number field K has positive analytic density if the abscissa of convergence of

$\prod_{\mathfrak{p} \in T} (1 - |O/\mathfrak{p}|^{-s})^{-1}$ is equal to 1. Any co-finite set of primes has positive analytic density.

Theorem 2.8. *Let K be a number field with ring of integers O_K , and let \mathbf{G} be a semi-simple algebraic group scheme defined over O_K . There are $C \in \mathbb{R}$ and $b_1, \dots, b_n \in \mathcal{A}$ such that, for every finite extension L of K with ring of integers O_L ,*

1. *The set $T(L)$ of primes $\mathfrak{q} \in \text{Spec}(O_L)$ for which there is a subset $J = J(\mathfrak{q}) \subset \{1, \dots, n\}$ such that $\zeta_{G(O_L, \mathfrak{q})} - 1 \sim_C \xi_{\sum_{i \in J} b_i, |O_L/\mathfrak{q}|}$ is co-finite.*
2. *The set $R(L)$ of primes $\mathfrak{q} \in \text{Spec}(O_L)$ such that $\zeta_{G(O_L, \mathfrak{q})} - 1 \sim_C \xi_{\sum_1^n b_i, |O_L/\mathfrak{q}|}$ has positive analytic density.*

The proof of Theorem 2.8 is given at the end of Section 5. We now show how Theorem 2.8 implies Theorem 1.5.

Proof of Theorem 1.5. Let $C \in \mathbb{R}$, $b_1, \dots, b_n \in \mathcal{A}$, and $T(L), R(L) \subset \text{Spec}(O_L)$ be as in Theorem 2.8, and set $b = b_1 + \dots + b_n$. As noted before, Theorem 2.2 implies that $\alpha(\mathbf{G}(\widehat{O_L})) = \alpha(\prod_{\mathfrak{q} \in \text{Spec}(O_L)} \mathbf{G}(O_L, \mathfrak{q})) = \alpha(\prod_{\mathfrak{q} \in T(L)} \mathbf{G}(O_L, \mathfrak{q}))$. Since $R(L) \subset T(L)$,

$$\alpha \left(\prod_{\mathfrak{q} \in R(L)} \mathbf{G}(O_L, \mathfrak{q}) \right) \leq \alpha \left(\prod_{\mathfrak{q} \in T(L)} \mathbf{G}(O_L, \mathfrak{q}) \right). \quad (4)$$

For $\mathfrak{q} \in R(L)$, we have $\zeta_{G(O_L, \mathfrak{q})} - 1 \sim_C \xi_{b, |O_L/\mathfrak{q}|}$. Hence, Lemma 2.5 implies that the left hand side of (4) is equal to the abscissa of convergence of $\prod_{\mathfrak{q} \in R(L)} (1 + \xi_{b, |O_L/\mathfrak{q}|})$. Similarly, since $\zeta_{G(O_L, \mathfrak{q})} - 1 \lesssim_C \xi_{b, |O_L/\mathfrak{q}|}$, for every $\mathfrak{q} \in T(L)$, the right hand side of (4) is less than or equal to the abscissa of convergence of $\prod_{\mathfrak{q} \in T(L)} (1 + \xi_{b, |O_L/\mathfrak{q}|})$. Since $R(L)$ has positive analytic density, these two abscissae are equal, and, consequently, $\alpha(\mathbf{G}(\widehat{O_L}))$ is equal to the abscissa of convergence of $\prod_{\mathfrak{q} \in \text{Spec}(O_L)} (1 + \xi_{b, |O_L/\mathfrak{q}|})$.

Similarly, $\alpha(\mathbf{G}(\widehat{O_K}))$ is equal to the abscissa of convergence of $\prod_{\mathfrak{p} \in \text{Spec}(O_K)} (1 + \xi_{b, |O_K/\mathfrak{p}|})$, but these two are equal, as follows from the following lemma:

Lemma 2.9. *Let $a \in \mathcal{A}$, and let K and L be number fields. Then the abscissae of convergence of $\prod_{\mathfrak{p} \in \text{Spec}(O_K)} (1 + \xi_{a, |O_K/\mathfrak{p}|})$ and $\prod_{\mathfrak{q} \in \text{Spec}(O_L)} (1 + \xi_{a, |O_L/\mathfrak{q}|})$ are equal.*

Proof. For any two sequences r_n, s_n of positive real numbers, the product $\prod_n (1 + r_n + s_n)$ converges if and only if the two products $\prod_n (1 + r_n)$ and $\prod_n (1 + s_n)$ converge. Thus, if $a = b + c$ in \mathcal{A} , then the abscissa of convergence of $\prod_{\mathfrak{p} \in \text{Spec}(O_K)} (1 + \xi_{a, |O_K/\mathfrak{p}|})$ is the maximum

of the abscissae of convergence of $\prod_{\mathfrak{p} \in \text{Spec}(O_K)} (1 + \xi_{b,|O_K/\mathfrak{p}|})$ and $\prod_{\mathfrak{p} \in \text{Spec}(O_K)} (1 + \xi_{c,|O_K/\mathfrak{p}|})$, and, similarly, for L . Thus we can assume that $a = \{(m, n)\}$ is a singleton.

If $n = 0$, neither product converges. If $n \neq 0$, let $\zeta_K(s)$ be the Dedekind zeta function of K . A simple computation shows that

$$\frac{\zeta_K(ns - m)}{\zeta_K(2(ns - m))} = \prod_{\mathfrak{p} \in \text{Spec}(O_K)} (1 + \xi_{c,|O_K/\mathfrak{p}|}(s)).$$

Since the abscissa of convergence of $\zeta_K(s)$ is equal to 1 and $\zeta_K(s) \neq 0$ for $s > 1$, we get that the abscissa of convergence of $\prod_{\mathfrak{p} \in \text{Spec}(O_K)} (1 + \xi_{b,|O_K/\mathfrak{p}|})$ is equal to $\frac{m+1}{n}$. The same argument shows that the abscissa of convergence of $\prod_{\mathfrak{q} \in \text{Spec}(O_L)} (1 + \xi_{b,|O_L/\mathfrak{q}|})$ is also $\frac{m+1}{n}$. \square

\square

3 Preliminaries

3.1 Relative Zeta Functions

Throughout this section, all groups G are such that $R_n(G)$ is finite for every positive integer n .

Definition 3.1. Suppose that G is a group, that $H \subset G$ is a normal subgroup, and that ρ is an irreducible, finite-dimensional representation of H . Denote the set of all finite-dimensional irreducible representations of G whose restriction to H contain ρ by $\text{Irr}(G|\rho)$. The relative zeta function of G over ρ is the generating function

$$\zeta_{G|\rho}(s) = \sum_{\theta \in \text{Irr}(G|\rho)} \left(\frac{\dim \theta}{\dim \rho} \right)^{-s} \quad (s \in \mathbb{C}).$$

Similarly, denote the number of representations $\theta \in \text{Irr}(G|\rho)$ such that $\dim \theta \leq n \cdot \dim \rho$ by $R_n(G|\rho)$.

Lemma 3.2. Let $N \subset H \subset G$ be groups. Assume that H is of finite index in G and that N is normal in G . Let $\tau \in \text{Irr}(N)$. Then for each $m \in \mathbb{N}$,

$$\frac{1}{[G : H]} \cdot R_{m/[G:H]}(H|\tau) \leq R_m(G|\tau) \leq [G : H] \cdot R_m(H|\tau)$$

and, for every $s \in \mathbb{R}$, if one of $\zeta_{H|\tau}(s)$ or $\zeta_{G|\tau}(s)$ converges, then so does the other, and

$$[G : H]^{-1-s} \zeta_{H|\tau}(s) \leq \zeta_{G|\tau}(s) \leq [G : H] \cdot \zeta_{H|\tau}(s).$$

Proof. Consider the bipartite graph whose vertices are $\text{Irr}(G|\tau) \sqcup \text{Irr}(H|\tau)$ and there is an edge between $\rho_1 \in \text{Irr}(G|\tau)$ and $\rho_2 \in \text{Irr}(H|\tau)$ if ρ_2 is a sub-representation of $\text{Res}_H^G \rho_1$. Note that

1. The degree of every vertex is positive and bounded by $[G : H]$.
2. If $\rho_1 \in \text{Irr}(G|\tau)$ and $\rho_2 \in \text{Irr}(H|\tau)$ are connected by an edge, then $\dim \rho_2 \leq \dim \rho_1 \leq [G : H] \cdot \dim \rho_2$.

Let $\text{Irr}(G|\tau)_m \subset \text{Irr}(G|\tau)$ be the set of representations of dimension less than or equal to $m \dim \tau$, and define similarly the set $\text{Irr}(H|\tau)_m$. The set $\text{Irr}(G|\tau)_m$ is contained in the set of neighbors of $\text{Irr}(H|\tau)_m$, so

$$|\text{Irr}(G|\tau)_m| \leq |\text{Irr}(H|\tau)_m| \cdot [G : H].$$

Similarly, the set $\text{Irr}(H|\tau)_{m/[G:H]}$ is contained in the set of neighbors of $\text{Irr}(G|\tau)_m$, so

$$|\text{Irr}(H|\tau)_{m/[G:H]}| \leq |\text{Irr}(G|\tau)_m| \cdot [G : H].$$

This proves the first two inequalities. Similar argument shows the other two. \square

Suppose that $N \subset G$ is a normal subgroup, and that ρ is a representation of N which is G -invariant. It is usually not true that the relative zeta function $\zeta_{G|\rho}$ is equal to the zeta function of the quotient G/N ; any non-abelian, step-2 nilpotent group and its center will give a counter-example. We describe now a situation in which this does hold. Recall that ρ defines an element in the second cohomology group of G/N with values in \mathbb{C}^\times , also known as the Schur multiplier of G/N . The construction is as follows; see [20, Chapter 11]. Suppose $\rho : N \rightarrow \text{GL}_d(\mathbb{C})$. Pick a coset representative $\tilde{a} \in G$ for every element a of G/N such that $\tilde{1} = 1$. For every $a \in G/N$, the representations ρ and $\rho^{\tilde{a}}$ are equivalent, and we choose $T_a \in \text{GL}_d(\mathbb{C})$ such that $T_a \rho T_a^{-1} = \rho^{\tilde{a}}$; for T_1 we choose the identity. Then one checks that, for all $a, b \in G/N$, the transformation $T_{ab}^{-1} T_a T_b \rho \left((\tilde{ab})^{-1} \tilde{ab} \right)$ commutes with ρ and thus defines a scalar $\beta(a, b) \in \mathbb{C}^\times$. This β is a cocycle representing the cohomology class associated to G , N , and ρ , which is independent of the choices involved.

Lemma 3.3. *Let G be a group with a finite-index normal subgroup $N \triangleleft G$, and let $\rho \in \text{Irr}(N)$ be a G -invariant representation of N . If the cohomology class that ρ defines vanishes, then $\zeta_{G|\rho}(s) = \zeta_{G/N}(s)$.*

Proof. By [20, Theorem 11.7], the vanishing of the cohomology class implies that ρ can be extended to a representation $\tilde{\rho}$ of G . By [20, Theorem 6.16], the map $\text{Irr}(G/N) \rightarrow \text{Irr}(G|\rho)$ given by $\tau \mapsto \tau \otimes \tilde{\rho}$ is a bijection. The claim of the lemma follows from this. \square

Lemma 3.4. *Let p be a prime. Let G be a pro-finite group with a finite-index normal pro- p group N , and let $\rho \in \text{Irr}(N)$ be G -invariant. Then the cohomology class in $H^2(G/N, \mathbb{C}^\times)$ associated to (G, N, ρ) has order a power of p .*

Proof. The dimension of ρ is a power of p , and, for every $h \in N$, the scalar $\det(\rho(h))$ is a p^n th root of unity, for some n . For every $a, b \in G/N$, taking determinants in the definition of β , we get $\beta(a, b)^{\dim \rho} = \det \left(T_{ab}^{-1} T_a T_b \rho \left((\tilde{a}b)^{-1} \tilde{a}b \right) \right)$. Since we are free to arrange $\det(T_x) = 1$ for all x , we get that $\beta(a, b)$ is a root of unity of order a power of p . \square

Lemma 3.5. *Let p be a prime number. Suppose that G is a finite group, and that $N \subset G$ is a central subgroup such that $(|N|, p) = 1$. Then $(|H^2(G, \mathbb{C}^\times)|, p) = 1$ if and only if $(|H^2(G/N, \mathbb{C}^\times)|, p) = 1$.*

Proof. Let $(E_n^{p,q})$ be the Lyndon–Hochschild–Serre spectral sequence associated to the central extension $1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1$. Since the order of N is prime to p , so are the orders of $H^1(N, \mathbb{C}^\times)$ and $H^2(N, \mathbb{C}^\times)$. Therefore, the orders of

$$E_2^{0,2} = H^0(G/N, H^2(N, \mathbb{C}^\times)) = H^2(N, \mathbb{C}^\times)$$

and

$$E_2^{1,1} = H^1(G/N, H^1(N, \mathbb{C}^\times)) = \text{Hom}(G/N, H^1(N, \mathbb{C}^\times))$$

are prime to p , and, hence, so are the orders of $E_\infty^{0,2}$ and $E_\infty^{1,1}$. The fact that $E_\infty^{p,q}$ converges to $H^*(G, \mathbb{C}^\times)$ implies that $|H^2(G, \mathbb{C}^\times)| = |E_\infty^{0,2}| \cdot |E_\infty^{1,1}| \cdot |E_\infty^{2,0}|$, so $|H^2(G, \mathbb{C}^\times)|$ is prime to p if and only if $|E_\infty^{2,0}|$ is prime to p .

Since the order of $E_2^{0,1} = H^0(G/N, H^1(N, \mathbb{C}^\times)) = H^1(N, \mathbb{C}^\times)$ is prime to p , we get that the order of $E_\infty^{2,0} = E_3^{2,0}$ is prime to p if and only if the order of $E_2^{2,0} = H^2(G, \mathbb{C}^\times)$ is prime to p , yielding the result. \square

Lemma 3.6. *For every root system Φ there is a constant C such that, for any finite field \mathbb{F}_q of characteristic p greater than C and for every connected reductive \mathbb{F}_q -algebraic group \mathbf{G} with root system Φ , the size of $H^2(\mathbf{G}(\mathbb{F}_q), \mathbb{C}^\times)$ is prime to q .*

Proof. Let \mathbb{F}_q be a finite field of characteristic p and let \mathbf{G} be a connected reductive \mathbb{F}_q -algebraic group with root system Φ .

Assume first that \mathbf{G} is semi-simple. Then there are almost-simple groups G_1, \dots, G_n such that $\mathbf{G}(\mathbb{F}_q)$ is a quotient of $G_1 \times \dots \times G_n$ by a central subgroup, and both n and the ranks of the groups G_i are bounded by some function of Φ . In particular, the size of the kernel of the quotient is bounded as a function of Φ . It is known (see, for example

[2, Table 5]) that there is a constant C_1 , depending only on Φ , such that the sizes of $H^1(G_i, \mathbb{C}^\times)$ and $H^2(G_i, \mathbb{C}^\times)$ are bounded by C_1 . By the Künneth formula, the sizes of $H^1(\prod G_i, \mathbb{C}^\times)$ and $H^2(\prod G_i, \mathbb{C}^\times)$ are bounded by some constant, C_2 , depending only on Φ . In particular, if p is greater than C_2 , then the size of $H^2(\prod G_i, \mathbb{C}^\times)$ is prime to q . By Lemma 3.5, the same is true for the size of $H^2(\mathbf{G}(\mathbb{F}_q), \mathbb{C}^\times)$ if p is larger than the size of the kernel of the quotient $\prod G_i \rightarrow \mathbf{G}(\mathbb{F}_q)$.

Now assume that \mathbf{G} is merely reductive. Let $\mathbf{S} = [\mathbf{G}, \mathbf{G}]$ be the derived subgroup of \mathbf{G} and let $\mathbf{T} = Z(\mathbf{G})^0$. It is well known that \mathbf{T} is a torus, \mathbf{S} is semi-simple, and $\mathbf{G}(\mathbb{F}_q)$ is a quotient of $\mathbf{T}(\mathbb{F}_q) \times \mathbf{S}(\mathbb{F}_q)$ by a central subgroup, whose size is bounded by some constant, C_3 , depending only on Φ . As shown above, if p is large enough, then the size of $H^2(\mathbf{S}(\mathbb{F}_q), \mathbb{C}^\times)$ is prime to q ; a similar claim for $H^1(\mathbf{S}(\mathbb{F}_q), \mathbb{C}^\times)$ also holds. Since the size of $\mathbf{T}(\mathbb{F}_q)$ is prime to q , so are the orders of its first and second cohomology groups. By the Künneth formula, the size of $H^2(\mathbf{T}(\mathbb{F}_q) \times \mathbf{S}(\mathbb{F}_q), \mathbb{C}^\times)$ is prime to q . By Lemma 3.5, if we assume, in addition, that p is larger than the size of the kernel of $\mathbf{T}(\mathbb{F}_q) \times \mathbf{S}(\mathbb{F}_q) \rightarrow \mathbf{G}(\mathbb{F}_q)$, then the size of $H^2(\mathbf{G}(\mathbb{F}_q), \mathbb{C}^\times)$ is prime to q . \square

3.2 Orbit Method

All pro- p groups in this paper will arise as open subgroups of $\mathbf{G}(O_{L,\mathfrak{q}})$, where O_L the ring of integers in a number field L , $O_{L,\mathfrak{q}}$ is the completion of O_L in a prime \mathfrak{q} , and \mathbf{G} is an algebraic group scheme over O_L that is smooth over $\text{Spec } O_L$. We denote the Lie algebra of \mathbf{G} by \mathfrak{g} .

Definition 3.7. *Let \mathfrak{q} be a finite prime of L . We say that a pro- p subgroup $H \subset \mathbf{G}(O_{L,\mathfrak{q}})$ is good if the following two conditions hold.*

1. *The logarithm series*

$$\log(X) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(X-1)^n}{n} = (X-1) - \frac{(X-1)^2}{2} + \dots$$

converges on H , setting up an injective map $\log: H \rightarrow \mathfrak{g}(O_{L,\mathfrak{q}})$; the exponential series

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} = 1 + X + \frac{X^2}{2!} + \dots$$

converges on $\log(H)$ and yields the inverse map $\exp: \log(H) \rightarrow H$.

2. The image $\log(H)$ is closed under addition and the Lie bracket, thus forming a \mathbb{Z}_p -Lie lattice. It is also closed under the adjoint action of H . For all $A, B \in \log(H)$, the Campbell–Hausdorff formula

$$\log(\exp(A) \cdot \exp(B)) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{r_i + s_i > 0} \frac{(\sum_{i=1}^m (r_i + s_i))^{-1}}{r_1! \cdot s_1! \cdot \dots \cdot r_m! \cdot s_m!} R_{r_1, s_1, \dots, r_m, s_m}(A, B)$$

holds, where the Lie polynomials $R_{r_1, s_1, \dots, r_m, s_m}(A, B)$ are defined by

$$R_{r_1, s_1, \dots, r_m, s_m}(A, B) = \begin{cases} \text{ad}(A)^{r_1} \text{ad}(B)^{s_1} \dots \text{ad}(A)^{r_m}(B) & \text{if } s_m = 1, \\ \text{ad}(A)^{r_1} \text{ad}(B)^{s_1} \dots \text{ad}(B)^{r_{m-1}}(A) & \text{if } r_m = 1, s_m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.8. *Let $\mathbf{G} \subset \text{GL}_N$ be an algebraic group scheme over the ring of integers O_L of a number field L . Let \mathfrak{q} be a prime of O_L extending a rational prime p satisfying $p > [L : \mathbb{Q}]N^2$. Then every pro- p subgroup $H \subset \mathbf{G}(O_{L, \mathfrak{q}})$ is good.*

Proof. Pro- p groups which are saturable in the sense of Lazard – for recent characterizations see [22, 17] – are good in the sense of Definition 3.7. The assertion thus follows from [22, Corollary 1.5] which implies that every pro- p subgroup $H \subset \text{GL}_N(O_{L, \mathfrak{q}})$ is saturable. \square

A good pro- p group H acts on $\mathfrak{h} = \log(H)$ via the adjoint action $\text{Ad}: H \rightarrow \text{Aut}(\mathfrak{h})$. Thus $\text{Ad}(h)(A)$ denotes the image of $A \in \mathfrak{h}$ under the adjoint action of $h \in H$. The adjoint action also induces an action, called the co-adjoint action, of H on the Pontryagin dual $\mathfrak{h}^\vee = \text{Hom}_{\text{cont}}(\mathfrak{h}, \mathbb{C}^\times)$, consisting of continuous homomorphisms from the abelian pro- p group \mathfrak{h} to \mathbb{C}^\times . Concretely, for $h \in H$, $A \in \mathfrak{h}$ and $\theta: \mathfrak{h} \rightarrow \mathbb{C}^\times$, one defines

$$(\text{Ad}^*(h)\theta)(A) = \theta(\text{Ad}(h^{-1})(A)).$$

The Kirillov orbit method for p -adic analytic pro- p groups, as described in [18] or [16], in conjunction with [17, Theorem A], yields the following proposition.

Proposition 3.9. *Let \mathbf{G} be an affine group scheme over O_L . There is a finite set S of primes of O_L such that, for all finite primes $\mathfrak{q} \notin S$, every pro- p subgroup $H \subset \mathbf{G}(O_{L, \mathfrak{q}})$ is good and, setting $\mathfrak{h} = \log(H)$, the following hold.*

1. *There is a function $\Omega: \mathfrak{h}^\vee \rightarrow \text{Irr}(H)$ which is constant on co-adjoint orbits and induces a bijection between the set of co-adjoint orbits in \mathfrak{h}^\vee and the set of irreducible characters of H .*

2. For $\theta \in \mathfrak{h}^\vee$, the character $\Omega(\theta)$ is given by

$$\Omega(\theta)(h) = \frac{1}{|\mathrm{Ad}^*(H)(\theta)|^{1/2}} \sum_{\phi \in \mathrm{Ad}^*(H)(\theta)} \phi(\log(h)) \quad (h \in H).$$

In particular, the degree of $\Omega(\theta)$ is $\Omega(\theta)(1) = |\mathrm{Ad}^*(H)(\theta)|^{1/2}$.

3. If $g \in \mathbf{G}(O_{L,\mathfrak{q}})$ normalizes H , then it normalizes \mathfrak{h} and $\Omega(\theta)^g = \Omega(\mathrm{Ad}^*(g^{-1})(\theta))$.

Proof. Suppose that $\mathbf{G} \hookrightarrow \mathrm{GL}_N$, as above. Then we can take S to be the set of all finite primes \mathfrak{q} of O_L extending a rational prime p satisfying $p \leq [L : \mathbb{Q}]N^2$. Noting that saturable pro- p groups of dimension at most p are potent, the assertions follow from [17, Theorem A] and [16, Theorem 5.2]. \square

Lemma 3.10. *Suppose G is a pro-finite group, containing open normal subgroups $K \subset H \subset G$. Suppose that H and K are good pro- p groups with Lie lattices $\mathfrak{h} = \log(H)$ and $\mathfrak{k} = \log(K)$, and whose irreducible characters are described by the orbit method. Then*

$$\zeta_G(s) = \sum_{\theta \in \mathfrak{h}^\vee} \frac{1}{|\theta^G|} (\dim \Omega(\theta))^{-s} \zeta_{G|\Omega(\theta)}(s).$$

Furthermore, if $\tau \in \mathfrak{k}^\vee$, then

$$\zeta_{G|\Omega(\tau)}(s) = \sum_{\theta \in \mathfrak{h}^\vee, \theta|_{\mathfrak{k}} = \tau} \frac{|\tau^H|}{|\theta^G|} \left(\frac{\dim \Omega(\theta)}{\dim \Omega(\tau)} \right)^{-s} \zeta_{G|\Omega(\theta)}(s).$$

Proof. For each $\sigma \in \mathrm{Irr}(H)$, the set $\mathrm{Irr}(G|\sigma)$ depends only on σ^G , and these sets form a partition of $\mathrm{Irr}(G)$. If we choose representatives σ_i for the G -orbits, we get

$$\zeta_G(s) = \sum_i (\dim \sigma_i)^{-s} \zeta_{G|\sigma_i}(s) = \sum_{\sigma \in \mathrm{Irr}(H)} \frac{1}{|\sigma^G|} (\dim \sigma)^{-s} \zeta_{G|\sigma}(s).$$

Consider the Orbit Method map $\Omega : \mathfrak{h}^\vee \rightarrow \mathrm{Irr}(H)$. Its fibers are the H -coadjoint orbits. Moreover, it is G -equivariant, so the pre-images of the G -orbits in $\mathrm{Irr}(H)$ are the G -orbits in \mathfrak{h}^\vee . Hence,

$$\begin{aligned} \zeta_G(s) &= \sum_{\sigma \in \mathrm{Irr}(H)} \frac{1}{|\sigma^G|} (\dim \sigma)^{-s} \zeta_{G|\sigma}(s) = \sum_{\theta \in \mathfrak{h}^\vee} \frac{1}{|\theta^H| \cdot |\Omega(\theta)^G|} (\dim \Omega(\theta))^{-s} \zeta_{G|\Omega(\theta)}(s) = \\ &= \sum_{\theta \in \mathfrak{h}^\vee} \frac{1}{|\theta^G|} (\dim \Omega(\theta))^{-s} \zeta_{G|\Omega(\theta)}(s). \end{aligned}$$

The proof of the second statement is similar, using the following consequence of Proposition 3.9: for $\tau \in \mathfrak{k}^\vee$ and $\theta \in \mathfrak{h}^\vee$, the character $\Omega(\tau)$ is a constituent of the restriction of $\Omega(\theta)$ to K if and only if $\theta|_{\mathfrak{k}} = \tau^h$ for a suitable $h \in H$. \square

3.3 Definable and Quantifier-Free Functions

We will use several notions from Model Theory, which we summarize below. For more details, we refer to [6]. Fix a fixed first-order language and a theory T in that language. Two formulas $\phi(x)$ and $\psi(x)$ —here and in the following x denotes tuple of variables—are called equivalent if the statement $(\forall x)\phi(x) \leftrightarrow \psi(x)$ belongs to T . A definable set is an equivalence class of formulas under this equivalence relation. We will say that a definable set X is (equivalent to) a quantifier-free definable set if there is a formula in the equivalence class X which does not contain quantifiers. If X is a definable set given by the formula $\phi(x_1, \dots, x_n)$ and M is a model of T , we denote the set of tuples $(a_1, \dots, a_n) \in M^n$ such that the formula $\phi(a_1, \dots, a_n)$ holds in M by $X(M)$. We sometimes write $x \in X$ instead of $x \in X(M)$ for some model M of T .

Example 3.11. *We consider the first order language of rings, and the theory of fields.*

1. *Any affine scheme X over \mathbb{Z} can be considered as a quantifier-free definable set. This means that there is a quantifier-free formula Y such that, for every field k , we have $X(k) = Y(k)$.*
2. *The definable set Y defined by the formula $\phi(x) := (\exists y)y^2 = x$ is not quantifier-free. Indeed, for any quantifier-free definable set $X \subset \mathbb{A}^1$, there is a constant C such that one of $|X(\mathbb{F}_p)|$ or $|\mathbb{F}_p \setminus X(\mathbb{F}_p)|$ is bounded by C , whereas $|Y(\mathbb{F}_p)| = \frac{p+1}{2}$ if $p > 2$.*

In fact, for the theory of fields, the quantifier-free definable sets are exactly the Boolean combinations of affine varieties.

Suppose that X and Y are definable sets in the same number of variables, given by the formulas ϕ and ψ respectively. We say that X is contained in Y if the statement $(\forall x)\phi(x) \rightarrow \psi(x)$ belongs to T . We define the operations \cap, \cup, \times on definable sets in the obvious way.

Suppose X, Y are definable sets. A definable set $Z \subset X \times Y$ is called a definable function, if, for all models M of T , the set $Z(M)$ is the graph of a function from $X(M)$ to $Y(M)$. In some occasions, if $f : X \rightarrow Y$ is a definable function, we will say that X is a definable family of (definable) sets with base Y . For $y \in Y$, we denote the fiber $f^{-1}(y)$ by X_y . It is a definable set in the enrichment of the language obtained by adding the coordinates of y as constants.

We show, for example, how projective spaces arise as definable sets in the theory of fields (perhaps enriched with constants in some ring). Let V be a vector space with a basis e_1, \dots, e_n ; we identify V with \mathbb{A}^n . The definable set $P \subset \mathbb{A}^n$, which is the disjoint union of $\{1\} \times \mathbb{A}^{n-1}, \{(0, 1)\} \times \mathbb{A}^{n-2}, \dots, \{(0, 0, \dots, 1)\} \times \mathbb{A}^0$, can function as the projective

space of V in the category of definable sets, in the following sense. Consider the definable family $U \subset P \times V$ such that $((a_1, \dots, a_n), (v_1, \dots, v_n)) \in U$ if and only if the determinants of all minors $\begin{pmatrix} a_i & a_j \\ v_i & v_j \end{pmatrix}$ vanish. For any definable set S and any definable set $X \subset S \times V$ with the property that X_s is a line in V , for any $s \in S$, there is a unique definable map $f : S \rightarrow P$ such that the pull-back of U via f is equal to X . Choosing a different (definable) basis, we get a different universal family over P , but there is a quantifier-free definable map from P to itself that interchanges the two universal families. In a similar way, there are quantifier-free definable sets that function as the Grassmannians of d -dimensional subspaces in V . We denote the union of all Grassmannians of V by $\text{Gr}(V)$.

Definition 3.12. *Let \mathfrak{g} be a Lie algebra scheme over a ring k . In the theory of fields enriched with constants from k , denote the definable subset of $\text{Gr}(\mathfrak{g})$ consisting of all Lie subalgebras of \mathfrak{g} by $\text{Grass}(\mathfrak{g})$, and denote the definable subset of $\text{Grass}(\mathfrak{g})$ consisting of all nilpotent subalgebras by $\text{Grass}(\mathfrak{g})^{\text{nilp}}$.*

Proposition 3.13. *Let \mathfrak{g} be a Lie algebra over a ring k . The following holds*

1. $\text{Grass}(\mathfrak{g})$ and $\text{Grass}(\mathfrak{g})^{\text{nilp}}$ are Zariski closed sets, and, in particular, quantifier-free definable.
2. The map $\text{Grass}(\mathfrak{g}) \mapsto \text{Grass}(\mathfrak{g})^{\text{nilp}}$ taking a Lie algebra to its unipotent radical (i.e. to the sub-algebra of unipotent elements in the solvable radical) is quantifier-free definable.
3. The map $\text{Grass}(\mathfrak{g})^{\text{nilp}} \mapsto \text{Grass}(\mathfrak{g})$ taking a Lie algebra to its normalizer is quantifier-free definable.
4. There is a quantifier-free definable function from $\text{Grass}(\mathfrak{g})$ to the set of all root systems of rank smaller than or equal to the rank of \mathfrak{g} , such that, for every Lie algebra $\mathfrak{h} \in \text{Grass}(\mathfrak{g})$, the value of the function at \mathfrak{h} is the root system of the semi-simplification of \mathfrak{h} (i.e. the quotient of \mathfrak{h} by its solvable radical).
5. Suppose that \mathfrak{g} is the Lie algebra of an algebraic group G . Then there is a quantifier-free definable subset of $\text{Grass}(\mathfrak{g}) \times G$ whose fiber over a Lie algebra \mathfrak{h} is the normalizer of \mathfrak{h} in G .

Proof. We prove 2. and 3. The other claims are easier. For 2., recall that an element $X \in \mathfrak{g}$ is in the solvable radical of a sub-algebra \mathfrak{h} if and only if the algebra generated by $[X, \mathfrak{h}]$ is solvable, which happens if and only if $w_{\dim \mathfrak{g}}(x_1, \dots, x_{2 \dim \mathfrak{g}}) = 1$ for all $x_i \in$

$[X, \mathfrak{h}]$, where the words w_i are defined recursively as follows: $w_1(x_1, x_2) = [x_1, x_2]$ and $w_{i+1}(x_1, \dots, x_{2^{i+1}}) = [w_i(x_1, \dots, x_{2^i}), w_i(x_{2^i+1}, \dots, x_{2^{i+1}})]$.

It follows that the collection of pairs $(\mathfrak{h}, X) \in \text{Grass}(\mathfrak{g}) \times \mathfrak{g}$ such that X is in the solvable radical of \mathfrak{h} is definable. Therefore, the collection of pairs (\mathfrak{h}, X) such that X is in the unipotent radical of \mathfrak{h} is definable, which implies that the collection of pairs $(\mathfrak{h}, \mathfrak{k}) \in \text{Grass}(\mathfrak{g}) \times \text{Grass}(\mathfrak{g})^{\text{nilp}}$ such that \mathfrak{k} is the unipotent radical of \mathfrak{h} is definable. Since this is the graph of the map in 2., we need only show that it is quantifier-free. Denote this set by A . For any field F , the set $A(F)$ is invariant under any automorphism of F . In particular, if F^a denotes the algebraic closure of F and \mathfrak{h} is subalgebra defined over F , then its unipotent radical is defined over F . It follows that $A(F) = A(F^a) \cap (\text{Grass}(\mathfrak{g}) \times \text{Grass}(\mathfrak{g})^{\text{nilp}})(F)$. By elimination of quantifiers over algebraically closed fields, there is a quantifier-free definable set B such that $A(F^a) = B(F^a)$. Therefore $A(F) = B(F^a) \cap (\text{Grass}(\mathfrak{g}) \times \text{Grass}(\mathfrak{g})^{\text{nilp}})(F) = B(F)$, which implies that A is quantifier-free.

Next we prove 3. Let $\text{Gr}(\mathfrak{g})$ be the Grassmannian of all subspaces in \mathfrak{g} . It is enough to show that the set of pairs $(U, V) \in \text{Gr}(\mathfrak{g})^2$ such that $U = \{u \in \mathfrak{g} | (\forall v \in V)[u, v] \in V\}$ is constructible. Moreover, it is enough to prove this Zariski locally in U and V . There exist neighborhoods \mathcal{U} of U and \mathcal{V} of V in $\text{Gr}(\mathfrak{g})$ such that

1. $\dim(W) = \dim(U) = a$ for every $W \in \mathcal{U}$.
2. $\dim(W) = \dim(V)$ for every $W \in \mathcal{V}$.
3. There are (polynomial) functions $x_1, \dots, x_{\dim \mathfrak{g}}$ from \mathcal{U} to \mathfrak{g} such that at each point W in \mathcal{U} , the vectors $x_i(W)$ form a basis of \mathfrak{g} , and $x_1(W), \dots, x_a(W)$ form a basis for W .
4. There is a trivialization y_1, \dots, y_b of the tangent bundle to $\text{Grass}(\mathfrak{g})$ over \mathcal{V} , where b is the dimension of the Grassmannian of $\dim V$ subspaces in \mathfrak{g} .

The action of \mathfrak{g} on $\text{Grass}(\mathfrak{g})$ coming from the adjoint action can be written in these coordinates as a matrix of polynomial functions $A_{i,j}$ where $i = 1, \dots, \dim \mathfrak{g}$, and $j = 1, \dots, b$. The condition $U = \{u \in \mathfrak{g} | (\forall v \in V)[u, v] \in V\}$ is equivalent to

1. $A_{i,j}(U, V) = 0$ for $i = 1, \dots, a$ and all j .
2. The submatrix $A_{i,j}(U, V)$, where $i = a + 1, \dots, \dim \mathfrak{g}$ and $j = 1, \dots, b$ has rank $\dim \mathfrak{g} - a$.

These conditions define a constructible set. □

The following proposition can be found, for example, in [7, Théorème 6.4] or [8, Main Theorem]

Proposition 3.14. *Suppose that $\phi(x, y)$ is a first-order formula in the language of rings. There is a constant C such that, for every finite field \mathbb{F}_q and every $a \in \mathbb{F}_q^m$, there is a natural number d such that the size of the set $\{x \in \mathbb{F}_q^n \mid \mathbb{F}_q \models \phi(x, a)\}$ is either 0, or between $\frac{1}{C}q^d$ and Cq^d .*

Lemma 3.15. *Let \mathbf{G} be an affine algebraic group over a finite field \mathbb{F}_q with at most C connected components, and let \mathfrak{g} be the Lie algebra of G . Then*

1. *There is a constant D_1 , depending only on C , such that, for every finite extension \mathbb{F}_{q^n} of \mathbb{F}_q , we have $\frac{1}{D_1}|\mathfrak{g}(\mathbb{F}_{q^n})| < |\mathbf{G}(\mathbb{F}_{q^n})| < D_1|\mathfrak{g}(\mathbb{F}_{q^n})|$.*
2. *Suppose that \mathbf{G} acts on a variety X in such a way that the stabilizer of any point in X has less than C connected components. There is a constant D_2 , depending only on C , such that, for every finite extension \mathbb{F}_{q^n} of \mathbb{F}_q and every $x \in X(\mathbb{F}_{q^n})$, we have*

$$\frac{1}{D_2} \frac{|\mathfrak{g}(\mathbb{F}_q)|}{|\mathfrak{h}(\mathbb{F}_q)|} \leq |\mathbf{G}(\mathbb{F}_q)x| \leq D_2 \frac{|\mathfrak{g}(\mathbb{F}_q)|}{|\mathfrak{h}(\mathbb{F}_q)|},$$

where \mathfrak{h} is Lie algebra of the stabilizer of x .

Proof. If \mathbf{G} is either a torus, a semi-simple group, or a unipotent group, then $\frac{1}{2^{\dim G}}q^{n \dim \mathbf{G}} \leq |\mathbf{G}(\mathbb{F}_{q^n})| \leq 2^{\dim G}q^{n \dim \mathbf{G}}$. This is easy to see for torus or a unipotent group, and, using the Bruhat decomposition, follows for a semisimple group. Therefore, the same estimate is true for any connected algebraic group. Finally, $|\mathbf{G}^0(\mathbb{F}_{q^n})| \leq |\mathbf{G}(\mathbb{F}_{q^n})| \leq |\mathbf{G}/\mathbf{G}^0| \cdot |\mathbf{G}^0(\mathbb{F}_{q^n})|$. This proves the first claim. The second claim follows from the first. \square

3.4 Valued Fields

We will use the Denef–Pas language of valued fields (see, for example [9, Section 2]). This is a first-order, three-sorted language. The three sorts are called the valued field sort, the residue field sort, and the value group sort. They are denoted by F , k , and Γ respectively. The function symbols are:

1. $+_{val}, \times_{val}$ from pairs of valued field sort variables to one valued field sort variable,
2. $+_{res}, \times_{res}$ from pairs of residue field sort variables to one residue field sort variable,

3. $+_{gr}$ from pairs of value group sort variables to one value group sort variable,
4. val from one valued field sort to one value group sort,
5. ac from one valued field sort to one residue field sort.

There is also one binary relation symbol, $<$, between two value group sort variables.

For us, the important structures for the language of valued fields will come from discrete valuation fields. Given a discrete valuation field E with a uniformizer ϖ , we interpret the valued field sort as E , the residue field sort as the residue field of E , and the value group sort as the value group of E (which is isomorphic to \mathbb{Z}). The functions $+_{val}, \times_{val}, +_{res}, \times_{res}, +_{gr}$ and the relation $<$ are interpreted as the usual operations and order. Finally, the function symbol val is interpreted as the valuation, and the function symbol ac is interpreted as

$$ac(x) = x\varpi^{-\text{val}(x)} \pmod{\varpi}.$$

The values of $val(0)$ and $ac(0)$ are irrelevant and will be chosen as ∞ . We will use the notion of dimension of a definable set, as it is defined, for example, in [9, Section 3].

The definable set $\{x \in F \mid \text{val}(x) \geq 0\}$ will be denoted by \mathcal{O} . For any discrete valuation field E , the set $\mathcal{O}(E)$ is the valuation ring O_E . Any O_E -scheme X gives rise to three definable sets in the language augmented with constants from O_E . Suppose that X is given by the vanishing of the polynomials $f_1(x), \dots, f_m(x) \in O_E[x]$. The first definable set is the set of all zeros of $f_i(x)$ in F^n ; we denote it by X_F . The second is the set of zeros of the reductions of $f_i(x)$ in k^n ; we denote it by X_k . The third is $X_{\mathcal{O}} = X_F \cap \mathcal{O}^n$. For example, for every local field E' that contains E , the set $G_{\mathcal{O}}(E')$ is equal to $G(O_{E'})$.

4 Parameterizing Representations

The goal of this section is to prove the following:

Theorem 4.1. *Suppose that $\mathbf{G} \subset \text{GL}_N$ is a semi-simple algebraic group defined over a number field K . There is a $(\dim(\mathbf{G}) + 1)$ -dimensional quantifier-free definable set $\mathcal{Y} \subset \mathcal{O}^{\dim(\mathbf{G})+1}$, quantifier-free definable functions $f_1, f_2 : \mathcal{Y} \rightarrow \Gamma$, and a constant C such that, for every finite field extension L/K and almost every prime ideal \mathfrak{q} of O_L ,*

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - 1 \sim_C \int_{\mathcal{Y}(L_{\mathfrak{q}})} |O_L/\mathfrak{q}|^{f_1(x)+f_2(x) \cdot s} d\lambda(x),$$

where λ is the normalized Haar measure on $O_{L,\mathfrak{q}}^{\dim(\mathbf{G})+1}$.

During the proof, there will be several places where we omit finitely many primes of O_L , or omit the primes of O_L lying over finitely many primes of O_K . We will write “for almost every prime” instead of “for all primes other than those omitted previously”. The primes we omit depend on L . However, the definable set \mathscr{Y} and the definable functions f_1, f_2 do not depend on the set of omitted primes.

4.1 Relative Orbit Method

Let $\mathbf{G} \subset \mathrm{GL}_n$ be a semi-simple algebraic group defined over a number field K , and let \mathfrak{g} be its Lie algebra. Fix a non-degenerate and Ad -invariant bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . There are finitely many primes of O_K for which the form $\langle \cdot, \cdot \rangle$ is non degenerate over $O_{K,\mathfrak{p}}$. We omit those, and the primes of L that lie over them. We consider \mathbf{G} and \mathfrak{g} as quantifier-free definable sets over the first-order language of valued fields together with constant symbols for the elements of K .

Definition 4.2. Let \mathscr{X} be the quantifier-free definable set $\mathscr{X} = \mathfrak{g}_{\mathcal{O}} \times (\mathcal{O} \setminus \{0\})$.

For every number field L containing K , and for almost every prime \mathfrak{q} of L , there is a surjective map $\Pi_{\mathfrak{q}} : \mathscr{X}(L_{\mathfrak{q}}) \rightarrow \mathfrak{g}(O_{L,\mathfrak{q}})^{\vee}$ taking the pair (A, z) to the character $\Pi_{\mathfrak{q}}(A, z) : \mathfrak{g}(O_{L,\mathfrak{q}}) \rightarrow \mathbb{C}^{\times}$ given by the formula

$$\Pi_{\mathfrak{q}}(A, z)(B) = \exp \left(2\pi i \cdot \mathrm{Tr}_{L_{\mathfrak{q}}/\mathbb{Q}_p} \left(\frac{\langle A, B \rangle}{z} \right) \right).$$

Omitting finitely many primes, we can assume that $\mathfrak{q} \cap O_K$ is unramified and Proposition 3.9 holds for \mathfrak{q} . By restricting $\Pi_{\mathfrak{q}}(A, z)$ to the first congruence subalgebra $\mathfrak{g}^{(1)}(O_{L,\mathfrak{q}})$, and applying the Orbit Method map, we get an irreducible representation of the group $\mathbf{G}^{(1)}(O_{L,\mathfrak{q}})$, which we denote by $\Xi_{\mathfrak{q}}(A, z)$. In our notation, $\Xi_{\mathfrak{q}}(A, z) = \Omega \left(\Pi_{\mathfrak{q}}(A, z)|_{\mathfrak{g}^{(1)}(O_{L,\mathfrak{q}})} \right)$. When the prime \mathfrak{q} is clear from the context, we will omit it from the notation.

For every finite extension L of K and every prime \mathfrak{q} of L , the set $\mathscr{X}(L_{\mathfrak{q}})$ is an open subset of $L_{\mathfrak{q}}^{\dim \mathbf{G} + 1}$. We normalize the Haar measure of $L_{\mathfrak{q}}$ so that the ring of integers has measure 1, and denote the restriction of the product measure on $L_{\mathfrak{q}}^{\dim \mathbf{G} + 1}$ to $\mathscr{X}(L_{\mathfrak{q}})$ by λ . In [21, Lemma 4.1 and Corollary 4.6], Jaikin–Zapirain proved the following:

Theorem 4.3. *There exist quantifier-free definable functions $f_1, f_2 : \mathscr{X} \rightarrow \Gamma$ such that, for every finite extension L of K , almost every prime \mathfrak{q} of L , and every $x \in \mathscr{X}(L_{\mathfrak{q}})$,*

1. $\lambda(\Pi_{\mathfrak{q}}^{-1}(\Pi_{\mathfrak{q}}(x))) = |O_L/\mathfrak{q}|^{f_1(x)}.$

$$2. \dim(\Xi_{\mathfrak{q}}(x)) = \left| \Pi_{\mathfrak{q}}(x) \mathbf{G}^1(O_{L,\mathfrak{q}}) \right|^{1/2} = |O_L/\mathfrak{q}|^{f_2(x)}.$$

Remark 4.4. In [21], Jaikin-Zapirain does not mention that the definable functions f_i are quantifier-free (and is actually working with a much bigger language), but they are given by a quantifier-free formula just before Corollary 4.6.

We will need a generalization of this construction. Employing the notation introduced in Section 3.4, the definable set $\text{Grass}(\mathfrak{g})_k$ is the Grassmannian of subalgebras of \mathfrak{g}_k , and the definable set $\text{Grass}(\mathfrak{g})_k^{\text{nilp}}$ is the subset of $\text{Grass}(\mathfrak{g})_k$ parameterizing nilpotent Lie subalgebras; according to Proposition 3.13, both are quantifier-free definable sets.

Suppose that $\mathcal{L} : \mathcal{X} \rightarrow \text{Grass}(\mathfrak{g})_k^{\text{nilp}}$ is a definable function. We define $\tilde{\mathcal{L}} \subset \mathcal{X} \times \mathfrak{g}_{\mathcal{O}}$ as the set of tuples (x, X) such that the reduction of X to \mathfrak{g}_k is in $\mathcal{L}(x)$. Omitting finitely many primes of O_K , we can assume that $N!$ is invertible in k (recall that $\mathbf{G} \subset \text{GL}_N$). We define $\exp \mathcal{L} \subset \mathcal{X} \times \mathbf{G}_k$ to be the set of pairs (x, g) such that g is unipotent and $\log(g - 1) \in \mathcal{L}(x)$. Finally, if, we omit, in addition, finitely many primes, we can assume that $3N \text{val}_{\mathfrak{q}}(\text{char } k) < \text{char } k$, and we define $\exp \tilde{\mathcal{L}} \subset \mathcal{X} \times \mathbf{G}_{\mathcal{O}}$ to consist of pairs (x, g) such that the reduction of g to \mathbf{G}_k is unipotent and $\log(g - 1) \in \tilde{\mathcal{L}}_x$ (recall that $\tilde{\mathcal{L}}_x$ denotes the fiber of $\tilde{\mathcal{L}}$ at x). Similarly, if $\mathcal{S} \subset \mathcal{X} \times \mathbf{G}_k$ is a definable family over \mathcal{X} , let $\tilde{\mathcal{S}} \subset \mathcal{X} \times \mathbf{G}_{\mathcal{O}}$ be the definable set of all pairs (x, g) such that the reduction of g in \mathbf{G}_k is in \mathcal{S}_x .

For every number field L containing K , for almost all primes \mathfrak{q} of L , and for any $x \in \mathcal{X}(L_{\mathfrak{q}})$, the additive group $\tilde{\mathcal{L}}_x(L_{\mathfrak{q}})$ is closed under commutators and is the Lie ring of the pro- p group $\exp \tilde{\mathcal{L}}_x(L_{\mathfrak{q}})$. We have $\mathfrak{g}^{(1)}(O_{L,\mathfrak{q}}) \subset \tilde{\mathcal{L}}_x(L_{\mathfrak{q}}) \subset \mathfrak{g}(O_{L,\mathfrak{q}})$ and $\mathbf{G}^{(1)}(O_{L,\mathfrak{q}}) \subset \exp \tilde{\mathcal{L}}_x(L_{\mathfrak{q}}) \subset \mathbf{G}(O_{L,\mathfrak{q}})$. Denote the restriction of $\Pi_{\mathfrak{q}}(x)$ to $\tilde{\mathcal{L}}_x(L_{\mathfrak{q}})$ by $\Pi_{\mathcal{L},\mathfrak{q}}(x)$. For almost all primes we can apply the orbit method map to $\Pi_{\mathcal{L},\mathfrak{q}}(x)$ and get a representation of $\exp \tilde{\mathcal{L}}_x(L_{\mathfrak{q}})$, which we denote by $\Xi_{\mathcal{L},\mathfrak{q}}(x)$. Again, when \mathfrak{q} is clear from the context, we will omit it from the notation. Note that, if \mathcal{L} is the constant function with common value $\{0\}$, then $\Xi_{\{0\},\mathfrak{q}}(x)$ coincides with our previously defined $\Xi_{\mathfrak{q}}(x)$. We will be interested in extensions of $\Xi_{\mathcal{L},\mathfrak{q}}(x)$ to its stabilizer in the normalizer of $\exp \tilde{\mathcal{L}}_x(L_{\mathfrak{q}})$ in $\mathbf{G}(O_{L,\mathfrak{q}})$, also known as the inertia group.

4.2 The Stabilizer of $\Xi_{\mathcal{L}}$

We denote the reduction map from \mathcal{O}^m to k^m by red . Suppose that $\mathcal{L} : \mathcal{X} \rightarrow \text{Grass}(\mathfrak{g})_k^{\text{nilp}}$ is a quantifier-free definable function. By Proposition 3.13, there are quantifier-free definable sets $\mathcal{N} = \mathcal{N}_{\mathcal{L}} \subset \mathcal{X} \times \mathbf{G}_k$ and $\mathcal{M} = \mathcal{M}_{\mathcal{L}} \subset \mathcal{X} \times \mathfrak{g}_k$ whose fibers over a point $x = (A, z) \in \mathcal{X}$ are the stabilizers of $\mathcal{L}(x)$ under the adjoint actions in the group and the Lie algebra respectively.

By part (3) of Proposition 3.9, the stabilizer of $\Xi_{\mathcal{L},\mathfrak{q}}(x)$ is the product of $\exp\left(\tilde{\mathcal{L}}_x(L_{\mathfrak{q}})\right)$ and the stabilizer of $\Pi_{\mathcal{L},\mathfrak{q}}(x)$ (recall that $\Pi_{\mathcal{L},\mathfrak{q}}(x)$ is the restriction of $\Pi_{\mathfrak{q}}(x)$ to $\tilde{\mathcal{L}}_x(L_{\mathfrak{q}})$). Denoting the set of elements $g \in \mathbf{G}(O_{L,\mathfrak{q}})$ such that $\mathrm{Ad}(g)\left(\tilde{\mathcal{L}}_x(L_{\mathfrak{q}})\right) = \tilde{\mathcal{L}}_x(L_{\mathfrak{q}})$ by $N_{\mathbf{G}(O_{L,\mathfrak{q}})}\tilde{\mathcal{L}}_x(L_{\mathfrak{q}})$, we have

$$\mathrm{Stab}_{\mathbf{G}(O_{L,\mathfrak{q}})}(\Pi_{\mathcal{L},\mathfrak{q}}(x)) = \left\{ g \in N_{\mathbf{G}(O_{L,\mathfrak{q}})}\tilde{\mathcal{L}}_x(L_{\mathfrak{q}}) \mid \left(\forall Y \in \tilde{\mathcal{L}}_x(L_{\mathfrak{q}}) \right) \quad \Pi_{\mathfrak{q}}(x)(Y) = \Pi_{\mathfrak{q}}(x)(\mathrm{Ad}(g)Y) \right\}$$

Fix a prime \mathfrak{p} of O_K , different from the previously-omitted primes, and an element (A, z) of $\mathcal{X}(K_{\mathfrak{p}})$. Let $\mathcal{S} \subset G_{\mathcal{O}}$ be the definable set given by the formula

$$\phi(g) := \mathrm{red}(g) \in (\mathcal{N}_{\mathcal{L}})_{(A,z)} \wedge \left(\forall Z \in \tilde{\mathcal{L}}_{(A,z)} \right) \mathrm{val}(\langle A^g - A, Z \rangle) > \mathrm{val}(z).$$

We can choose a finite set $Z_1, \dots, Z_n \in \mathfrak{g}(O_{K,\mathfrak{p}})$ such that, for any extension $L_{\mathfrak{q}}$ of $K_{\mathfrak{p}}$, the Lie ring $\tilde{\mathcal{L}}_{(A,z)}(L_{\mathfrak{q}})$ is the $O_{L,\mathfrak{q}}$ -module generated by the Z_i . In particular, ϕ is equivalent to a quantifier-free formula.

We denote the reduction of \mathcal{S} modulo the maximal ideal, i.e., the definable subset of G_k given by the formula

$$\psi(y) := (\exists g \in \mathbf{G}_{\mathcal{O}}) (\mathrm{red}(g) = y \wedge \phi(g)),$$

by $\overline{\mathcal{S}}$.

Using Witt vectors, there is a pro-algebraic group scheme \underline{S} and a definable group \underline{R} over O_K/\mathfrak{p} (that depend on \mathcal{L} , and x) such that, for all unramified extensions $(L_{\mathfrak{q}}, \mathfrak{q})$ of $(K_{\mathfrak{p}}, \mathfrak{p})$, we have $\mathcal{S}(L_{\mathfrak{q}}) = \underline{S}(O_{L_{\mathfrak{q}}}/\mathfrak{q})$ and $\overline{\mathcal{S}}(L_{\mathfrak{q}}) = \underline{R}(O_{L_{\mathfrak{q}}}/\mathfrak{q})$.

Proposition 4.5. *Let p be the residue characteristic of $K_{\mathfrak{p}}$. If p is greater than its valuation in $K_{\mathfrak{p}}$, then \underline{R} is (equivalent to) an algebraic group.*

Proof. Let $k = O_K/\mathfrak{p}$ be the residue field of $K_{\mathfrak{p}}$, and denote the algebraic closure of k by k^a . We first show that if the characteristic of k is greater than the valuation of p then $\underline{R}(k) = \underline{R}(k^a) \cap \mathbf{G}(k)$. The inclusion \subset is clear. In the other direction, suppose that $A, B \in \mathfrak{g}(O_{K,\mathfrak{p}})$, that $\overline{g} \in \mathbf{G}(O_K/\mathfrak{p})$, and that $z \in O_{K_{\mathfrak{p}}} \setminus \{0\}$. Let $\gamma = \mathrm{val}(z)$, let $x = (A, z)$ and let $Y_{A,B,\mathcal{L}_x,\overline{g},\gamma}$ be the quotient of

$$\left\{ g \in \mathbf{G}_{\mathcal{O}} \mid \mathrm{red}(g) = \overline{g} \wedge \left(\forall X \in \tilde{\mathcal{L}}_x \right) (\mathrm{val}(\langle A^g - B, X \rangle) > \gamma) \right\}$$

by the $(\gamma + 1)$ st congruence subgroup of $\mathbf{G}_{\mathcal{O}}$. This is not a definable set (but rather an imaginary), but this is irrelevant for the argument. Using Witt vectors, $Y_{A,B,\mathcal{L}_x,\overline{g},\gamma}$ is

in bijection with an algebraic variety over k . We need to show that, if $Y_{A,A,\mathcal{L}_x,\bar{g},\gamma}(k^a)$ is non-empty, then $Y_{A,A,\mathcal{L}_x,\bar{g},\gamma}(k)$ is non-empty. We will show, more generally, that, for every A, B, x, \bar{g} , and γ , if $Y_{A,B,\mathcal{L}_x,\bar{g},\gamma}(k^a)$ is non-empty, then so is $Y_{A,B,\mathcal{L}_x,\bar{g},\gamma}(k)$.

Note that if $h \in \mathbf{G}_{\mathcal{O}}$, then $h^{-1}Y_{A,B,\mathcal{L}_x,\bar{g},\gamma} = Y_{A^h,B,\mathcal{L}_x,\overline{h^{-1}g},\gamma}$. Thus, we can assume that $\bar{g} = 1$.

Suppose first that $\mathcal{L}_x = \{0\}$. This means that $\tilde{\mathcal{L}}_x$ is the first congruence algebra. Since the form $\langle \cdot, \cdot \rangle$ is non-degenerate, the second condition in the definition of $Y_{A,B,\{0\},1,\gamma}$ is equivalent to $\text{val}(A^g - B) \geq \gamma$. Thus, $Y_{A,B,\{0\},1,\gamma}$ is a torsor of the subgroup $\Sigma_{A,\gamma}$ of all elements $g\mathbf{G}_{\mathcal{O}}^{(\gamma+1)}$ in $\mathbf{G}_{\mathcal{O}}^{(1)}/\mathbf{G}_{\mathcal{O}}^{(\gamma+1)}$ such that $\text{val}(A^g - A) \geq \gamma$. Using Witt vectors, we think about $\Sigma_{A,\gamma}$ as an algebraic group defined over k . Since this group is unipotent, it is enough to prove that it is connected (because connected unipotent groups have trivial first Galois cohomology groups, and, thus, every torsor over such a group has a point). Since we assume that the characteristic of k is large enough, the logarithm map is a well-defined polynomial map and gives a bijection between $\Sigma_{A,\gamma}$ and the sub-variety of $\mathfrak{g}_{\mathcal{O}}^1/\mathfrak{g}_{\mathcal{O}}^{\gamma}$ consisting of elements Z that satisfy $\text{val}([Z, A]) \geq \gamma$. The latter is an affine space and, hence, connected.

For general \mathcal{L}_x , note that if $Y_{A,B,\mathcal{L}_x,1,\gamma}(k^a)$ is non-empty, then $Y_{A,B,\{0\},1,\gamma-1}(k^a)$ is non-empty, and so, by the previous paragraph, there is $f \in \mathbf{G}^{(1)}(O_{K,\mathfrak{p}})$ such that $B^f = A + \pi^{\gamma}E$ for some $E \in \mathfrak{g}(O_{K,\mathfrak{p}})$. Since $Y_{A,B,\mathcal{L}_x,1,\gamma}f^{-1} = Y_{A,A+\pi^{\gamma}E,\mathcal{L}_x^f,1,\gamma}$, we can assume that $B = A + \pi^{\gamma}E$. Applying the logarithm map, the variety $Y_{A,A+\pi^{\gamma}E,\mathcal{L}_x,1,\gamma}$ is mapped to

$$\left\{ Z \in \mathfrak{g}_{\mathcal{O}}^{(1)}/\mathfrak{g}_{\mathcal{O}}^{(\gamma+1)} \mid \left(\forall X \in \tilde{\mathcal{L}}_x \right) (\text{val}(\langle [A, Z] - B, X \rangle) > \gamma) \right\}$$

which is an affine space defined over k , and therefore has a k -rational point. Thus, we have shown that if $Y_{A,B,\mathcal{L}_x,\bar{g},\gamma}$ is non-empty, then it contains a k -rational point. This concludes the proof that $\underline{R}(k) = \underline{R}(k^a) \cap \mathbf{G}(k)$.

The theory of algebraically closed fields admits elimination of quantifiers. Thus, $\underline{R}(k^a)$ is equal to the k^a -points of a quantifier-free definable set $W \subset \mathbf{G}_k$. By what we proved, $\underline{R}(k) = \underline{R}(k^a) \cap \mathbf{G}(k) = W(k^a) \cap \mathbf{G}(k) = W(k)$. Therefore \underline{R} is a quantifier-free definable subgroup of G_k , which must be Zariski closed. \square

Slightly abusing notation, we denote the algebraic group equivalent to \underline{R} again by \underline{R} , or, if we want to emphasize its dependence on \mathcal{L} and x , by $\underline{R}_{\mathcal{L},x}$.

Proposition 4.6. *There is a constant C (depending only on K and \mathbf{G}) such that, for any finite extension L of K , almost any prime \mathfrak{q} of O_L , any quantifier-free definable function $\mathcal{L} : \mathcal{X} \rightarrow \text{Grass}(\mathfrak{g})_k^{\text{nilp}}$, and every $x \in \mathcal{X}(L_{\mathfrak{q}})$, the number of connected components of $\underline{R}_{\mathcal{L},x}$ is less than C .*

Proof. By elimination of quantifiers in Henselian fields of residue characteristic 0 (see [19]), after omitting finitely many primes, the formula defining $\overline{\mathcal{S}}$ is equivalent to a formula of the form

$$\eta(x) := \bigvee_j \phi_j(\text{val}(f_i(A)), \gamma) \wedge \psi_j(\text{ac}(f_i(A)), \mathcal{L}, x),$$

where the f_i are polynomials in the entries of A , ϕ_j are formulas in the language of ordered groups, and ψ_j are formulas in the language of rings. It follows that \mathcal{S} is a union of some of the definable sets $\{x \mid \psi_j(\text{ac}(f_i(A)), \mathcal{L}, x)\}$. By Proposition 3.14, there is a constant C such that, for any unramified finite extension $M_{\mathfrak{r}}$ of $L_{\mathfrak{q}}$ with residue field \mathbb{F}_{q^r} , there is a natural number d such that $\frac{1}{C}q^{rd} \leq |\overline{\mathcal{S}}(M_{\mathfrak{r}})| = |\underline{R}(\mathbb{F}_{q^r})| \leq Cq^{rd}$. Since \underline{R} is an algebraic variety, there are infinitely many integers r such that all absolutely irreducible components of \underline{R} are defined over \mathbb{F}_{q^r} , and so $\underline{R}(\mathbb{F}_{q^r})$ contains approximately $|\underline{R}/\underline{R}^{\circ}|q^{r \dim \underline{R}}$ points. Comparing the two expressions as r tends to infinity, we get that $|\underline{R}/\underline{R}^{\circ}| \leq C$. \square

We now find the Lie algebra of \underline{R} . Let $\mathcal{T} \subset \mathfrak{g}_{\mathcal{O}}$ be the definable set given by the formula

$$\xi(X) := \text{red}(X) \in (\mathcal{M}_{\mathcal{L}})_{(A,z)} \wedge \left(\forall Z \in \tilde{\mathcal{L}}_{(A,z)} \right) \text{val}(\langle [A, X], Z \rangle) > \text{val}(z),$$

and let $\overline{\mathcal{T}}$ be the reduction of \mathcal{T} modulo the maximal ideal. The argument after the definition of \mathcal{S} that showed that \mathcal{S} is quantifier-free shows that \mathcal{T} is quantifier-free. Similarly to the proof of Proposition 4.5, for every L and almost any prime \mathfrak{q} of O_L , $\overline{\mathcal{T}}(L_{\mathfrak{q}})$ is a linear space over the residue field of $L_{\mathfrak{q}}$. We will give a different (and more explicit) proof of this in Subsection 4.3.

Proposition 4.7. *The Lie algebra of \underline{R} is equal to $\overline{\mathcal{T}}(K_{\mathfrak{p}})$.*

Proof. Suppose that $x = (A, z)$ and let $\gamma = \text{val}(z)$. Consider the quotient $\underline{S}^{(\gamma)} = \underline{S} / (\underline{S} \cap \mathbf{G}_{\mathcal{O}}^{(\gamma+1)})$. Then $\underline{S}^{(\gamma)}$ is an algebraic group, and the quotient map $f : \underline{S} \rightarrow \underline{R}$ factors through the quotient $f^{(\gamma)} : \underline{S}^{(\gamma)} \rightarrow \underline{R}$. By definition, this homomorphism is onto. Being an epimorphism between algebraic groups, it must be flat ([14, Proposition 6.1.5]). As shown in the proof of Proposition 4.5, the fibers of $f^{(\gamma)}$, each isomorphic to the kernel of $f^{(\gamma)}$, are affine spaces, and hence smooth. By [15, Theorem 17.5.1], the map $f^{(\gamma)}$ is smooth, and so its differential at 1 is surjective. It follows that the same is true for f . It is easy to see that the Lie algebra of \underline{S} is equal to $\mathcal{T}(K_{\mathfrak{p}})$, and its image under the reduction modulo the maximal ideal is $\overline{\mathcal{T}}(K_{\mathfrak{p}})$. \square

In conjunction with Lemma 3.15 we get

Corollary 4.8. *There is a constant C such that, for every extension L of K and almost all prime ideals \mathfrak{q} of L ,*

$$\frac{1}{C}|\overline{\mathcal{T}}(L_{\mathfrak{q}})| \leq |\overline{\mathcal{S}}(L_{\mathfrak{q}})| \leq C|\overline{\mathcal{T}}(L_{\mathfrak{q}})|.$$

4.3 The Lie Algebra of the Stabilizer of $\Xi_{\mathcal{L}}$

It would be easier, and more transparent, to work with the Lie algebra of the stabilizer of $\Xi_{\mathcal{L}}$, rather than with the stabilizer itself. For this, we introduce the following cover of \mathcal{X} :

Definition 4.9. *Let $\mathcal{Y} \subset \mathcal{X} \times (\Gamma \cup \{\infty\})^{\dim \mathfrak{g}} \times \text{Aut}(\mathfrak{g})_{\mathcal{O}} \times \text{Aut}(\mathfrak{g})_{\mathcal{O}}$ be the quantifier-free definable set consisting of tuples $(A, z, \gamma_1, \dots, \gamma_{\dim \mathfrak{g}}, U_1, U_2)$ such that, in the standard basis, the operator $U_1(\text{ad } A)U_2$ is diagonal, and the valuations of the diagonal elements are $\gamma_1, \dots, \gamma_{\dim \mathfrak{g}}$.*

We denote the projection from \mathcal{Y} to \mathcal{X} by pr . Shortening the notation, we denote the composition $\Pi_{\mathfrak{q}} \circ pr$ simply by $\Pi_{\mathfrak{q}}$. Given a definable map $\mathcal{L} : \mathcal{Y} \rightarrow \text{Grass}(\mathfrak{g})_k^{nilp}$, we define $\tilde{\mathcal{L}} \subset \mathcal{Y} \times \mathfrak{g}_{\mathcal{O}}$, $\exp \tilde{\mathcal{L}} \subset \mathcal{Y} \times G_{\mathcal{O}}$, and $\Xi_{\mathcal{L}, \mathfrak{q}}$ similarly as for \mathcal{X} .

The following lemma is evident:

Lemma 4.10. *Let O be a complete, discrete valuation ring with a uniformizer ϖ , let $M = O^n$, and let $\overline{N} \subset M/\varpi M$ be a linear subspace. Denote the pre-image of \overline{N} in M by N . Assume that T is an endomorphism of M which, in the standard basis, is given by a diagonal matrix with diagonal entries $\varpi^{\gamma_1}, \dots, \varpi^{\gamma_n}$ such that $\gamma_1 \leq \dots \leq \gamma_n$. For every l , let $i(l)$ be the maximal index for which $\gamma_i \leq l$ and $j(l)$ be the minimal index such that $\gamma_j \geq l$. Then,*

1. *The pre-image $T^{-1}(\varpi^l N)$ is equal to*

$$\left\{ (a_1, \dots, a_n) \mid \text{val}(a_1) \geq l - \gamma_1, \dots, \text{val}(a_{i(l)}) \geq l - \gamma_{i(l)} \text{ and } (\overline{\varpi^{\gamma_1 - l} a_1}, \dots, \overline{\varpi^{\gamma_{i(l)} - l} a_{i(l)}}, 0, \dots, 0) \in \overline{N} \right\}.$$

2. *The reduction of $T^{-1}(\varpi^l N)$ modulo ϖ is the set of all $(\overline{a_1}, \dots, \overline{a_n}) \in (O/\varpi)^n$ such that $\overline{a_1} = \dots = \overline{a_{j(l)-1}} = 0$ and $(0, \dots, 0, \overline{a_{j(l)}}, \dots, \overline{a_{i(l)}}, 0, \dots, 0) \in \overline{N}$.*

Proposition 4.11. *For every quantifier-free definable function $\mathcal{R} : \mathcal{Y} \rightarrow \text{Grass}(\mathfrak{g})_k^{nilp}$ there is a quantifier-free definable function $\mathcal{L} : \mathcal{Y} \rightarrow \text{Grass}(\mathfrak{g})_k$ such that, for every finite extension L of K , almost every prime \mathfrak{q} of L , and every $y \in \mathcal{Y}(L_{\mathfrak{q}})$, the Lie algebra of the stabilizer of $\Xi_{\mathcal{R}, \mathfrak{q}}(y)$ is equal to $\tilde{\mathcal{L}}_y$.*

Proof. Let $y = (A, z, \gamma_i, U_1, U_2)$. The Lie algebra of the stabilizer of $\Xi_{\mathcal{R}, \mathfrak{q}}(y)$ is the sum of $\tilde{\mathcal{R}}(y)$ and the Lie algebra of the stabilizer of $\Pi_{\mathcal{R}}(y)$. It is enough to prove that there is a quantifier-free definable function \mathcal{L} such that $\mathcal{L}(y)$ is the reduction of the Lie algebra of the stabilizer of $\Pi_{\mathcal{R}}(y)$. The latter is the intersection of the normalizer of $\mathcal{R}(y)$ (which is quantifier-free) with

$$\begin{aligned} V &= \left\{ Y \mid (\forall X \in \tilde{\mathcal{R}}(y)) \Pi_{\mathcal{R}}(y)([X, Y]) = 1 \right\} = \left\{ Y \mid (\forall X \in \tilde{\mathcal{R}}(y)) \text{val}(\langle A, [X, Y] \rangle) \geq \text{val}(z) \right\} = \\ &= \left\{ Y \mid (\forall X \in \tilde{\mathcal{R}}(y)) \text{val}(\langle X, [A, Y] \rangle) \geq \text{val}(z) \right\}. \end{aligned}$$

If we let $\mathcal{R}(y)^\perp$ be the orthogonal subspace to $\mathcal{R}(y)$ in \mathfrak{g} (with respect to the form $\langle \cdot, \cdot \rangle$), then V is the pre-image of $z\mathcal{R}(y)^\perp$ under the map $\text{ad}(A)$. Since $\text{ad}(A) = U_1(y)T(y)U_2(y)$, for $U_i(y) \in \text{Aut}(\mathfrak{g})_{\mathcal{O}}$ and $T(y)$ diagonal, $U_1(y)(V)$ is the pre-image of $zU_2(y)\mathcal{R}(y)^\perp$ under $T(y)$, which is defined by a quantifier-free formula thanks to the second statement of Lemma 4.10. \square

Corollary 4.12. *For every quantifier-free definable function $\mathcal{R} : \mathcal{X} \rightarrow \text{Grass}(\mathfrak{g})_k^{\text{nilp}}$, there is a quantifier-free definable function $\mathcal{L} : \mathcal{X} \rightarrow \text{Grass}(\mathfrak{g})_k$ such that for every finite extension L of K , almost every prime \mathfrak{q} of L , and every $x \in \mathcal{X}(L_{\mathfrak{q}})$, the Lie algebra of the stabilizer of $\Xi_{\mathcal{R}, \mathfrak{q}}(x)$ is equal to $\mathcal{L}(x)$.*

Proof. Pre-composing \mathcal{R} with the projection $pr : \mathcal{Y} \rightarrow \mathcal{X}$, and applying Proposition 4.11, we get a quantifier-free definable function $\mathcal{L}_1 : \mathcal{Y} \rightarrow \text{Grass}(\mathfrak{g})_k$ such that, for all $y \in \mathcal{Y}(L_{\mathfrak{q}})$, the vector space $\mathcal{L}_1(y)$ is the Lie algebra of the stabilizer of $\Xi_{\mathcal{R}_{\text{opr}}, \mathfrak{q}}(y)$. Since $\mathcal{L}_1(y)$ depends only on $pr(y)$, we get a definable function $\mathcal{L}_2 : \mathcal{X} \rightarrow \text{Grass}(\mathfrak{g})_k^{\text{nilp}}$ such that $\mathcal{L}_2(x)$ is the Lie algebra of the stabilizer of $\Xi_{\mathcal{R}, \mathfrak{q}}(x)$ for all $x \in \mathcal{X}(L_{\mathfrak{q}})$. By a well-known criterion for elimination of quantifiers (see, for example [27, Theorem 3.1.4]), in order to show that \mathcal{L}_2 is equal to a quantifier-free function, it is enough to show that if $F_1 \subset F_2$ is an extension of valued fields, $x \in \mathcal{X}(F_1)$, and $v \in \text{Grass}(\mathfrak{g})_k(F_1)$, then $F_1 \models \mathcal{L}_2(x) = v$ if and only if $F_2 \models \mathcal{L}_2(x) = v$. This is true because if $y \in \mathcal{Y}(F_1) \subset \mathcal{Y}(F_2)$ is such that $pr(y) = x$, then $F_1 \models \mathcal{L}_2(x) = v$ iff $F_1 \models \mathcal{L}_1(y) = v$ iff $F_2 \models \mathcal{L}_1(y) = v$ (because \mathcal{L}_1 is quantifier-free) iff $F_2 \models \mathcal{L}_2(x) = v$. \square

Proposition 4.13. *For every quantifier-free definable map $\mathcal{R} : \mathcal{X} \rightarrow \text{Grass}(\mathfrak{g})_k^{\text{nilp}}$, there is a quantifier-free definable function $f : \mathcal{X} \rightarrow \Gamma$ such that, for every finite extension L of K , almost every prime \mathfrak{q} of L , and every $x \in \mathcal{X}(L_{\mathfrak{q}})$,*

$$\left| \Pi_{\mathcal{R}, \mathfrak{q}}(x)^{\exp \tilde{\mathcal{R}}(x)} \right| = |O_L/\mathfrak{q}|^{f(x)}.$$

Proof. Similarly to the proof of Proposition 4.11, the first claim of Lemma 4.10 gives a quantifier-free function $f_1 : \mathcal{Y} \rightarrow \Gamma$ such that, for every $y \in \mathcal{Y}(L_{\mathfrak{q}})$,

$$\left| \Pi_{\mathcal{R} \circ pr, \mathfrak{q}}(y)^{\exp \tilde{\mathcal{R}}(pr(y))} \right| = |O_L/\mathfrak{q}|^{f_1(y)}.$$

Since $f_1(y)$ depends only on the image of y in \mathcal{X} , we get a definable function $f_2 : \mathcal{X} \rightarrow \Gamma$ such that

$$\left| \Pi_{\mathcal{R}, \mathfrak{q}}(x)^{\exp \tilde{\mathcal{R}}(x)} \right| = |O_L/\mathfrak{q}|^{f_2(x)}.$$

A similar argument to the one in Corollary 4.12 shows that f_2 is (equal to) a quantifier-free definable function. \square

Proposition 4.14. *There is a quantifier-free definable function $f : \mathcal{X} \rightarrow \Gamma$ such that, for every finite extension L of K , almost every prime \mathfrak{q} of L , and every $x \in \mathcal{X}(L_{\mathfrak{q}})$, the normalized Haar measure of the set $\Pi_{\{0\}}^{-1}(\Pi_{\{0\}}(x))$ is equal to $|O_L/\mathfrak{q}|^{f(x)}$.*

Proof. Suppose that $x = (A, z)$. Then $\Pi_{\{0\}}^{-1}(\Pi_{\{0\}}(x))$ consists of the pairs (B, w) such that, for all $X \in \mathfrak{g}^1(O_{L, \mathfrak{q}})$,

$$\text{val} \left(\left\langle \frac{A}{z} - \frac{B}{w}, X \right\rangle \right) > 0,$$

or, equivalently, such that $\text{val}(Aw - Bz) \geq \text{val}(zw)$. The proposition now follows. \square

4.4 Proof of Theorem 4.1

Theorem 4.15. *For every $n \geq 0$ there are quantifier-free definable functions $f_n, h_n : \mathcal{X} \rightarrow \Gamma$, $\mathcal{R}_n : \mathcal{X} \rightarrow \text{Grass}(\mathfrak{g})_k^{\text{nilp}}$, $\mathcal{L}_n : \mathcal{X} \rightarrow \text{Grass}(\mathfrak{g})_k$, a constant C_n , and a definable family $\mathcal{S}_n \subset \mathcal{X} \times \mathbf{G}_k$ of subgroups of \mathbf{G}_k such that*

1. \mathcal{R}_0 is the constant function $\{0\}$, \mathcal{L}_0 is the constant function \mathfrak{g} , and $\mathcal{S}_0 = \mathcal{X} \times \mathbf{G}_k$.
2. For every L, \mathfrak{q} and for every $x \in \mathcal{X}(L_{\mathfrak{q}})$, $(\widetilde{\mathcal{S}}_n)_x$ is the stabilizer of $\Xi_{\mathcal{R}_{n-1}}(x)$ in $(\widetilde{\mathcal{S}}_{n-1})_x$, $\mathcal{L}_n(x)$ is the Lie algebra of $(\mathcal{S}_n)_x$, and $\mathcal{R}_n(x)$ is the nilpotent radical of $\mathcal{L}_n(x)$.
3. There is N such that the sequences $f_n, h_n, \mathcal{R}_n, \mathcal{S}_n, \mathcal{L}_n$ stabilize for $n > N$.
4. For every n and L , for almost all \mathfrak{q} ,

$$\zeta_{\mathbf{G}(O_{L, \mathfrak{q}})}(s) - 1 \sim_{C_n} \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) - 1 + \int_{\mathcal{X}(L_{\mathfrak{q}})} |O_L/\mathfrak{q}|^{f_n(x) + h_n(x)s} \cdot \zeta_{(\widetilde{\mathcal{S}}_n)_x(L_{\mathfrak{q}}) | \Xi_{\mathcal{R}_n}(x)}(s) d\lambda(x).$$

Proof. We first construct $\mathcal{R}_n, \mathcal{S}_n, \mathcal{L}_n$ by induction. The case $n = 0$ is given by the first requirement. Given $\mathcal{R}_n, \mathcal{S}_n, \mathcal{L}_n$, the discussion in the beginning of Subsection 4.2 implies that there is a definable family of subgroups of $G_{\mathcal{O}}$ whose fiber at any x is the stabilizer of $\Xi_{\mathcal{R}_n}(x)$ in $(\widetilde{\mathcal{S}_n})_x$. Take \mathcal{S}_{n+1} to be the reduction of this family to k . Similarly, we get \mathcal{L}_{n+1} from Corollary 4.12 and \mathcal{R}_{n+1} from Lemma 3.13.

Next, we show that the sequences $\mathcal{R}_n, \mathcal{L}_n$, and \mathcal{S}_n stabilize. Note that the sequence $\dim(\mathcal{R}_n)$ is (pointwise) non-decreasing and the sequence $\dim(\mathcal{L}_n)$ is non-increasing. Proposition 3.14 implies that, for any n , there is a bound $D(n)$ on the number of connected components of any of the groups $(\mathcal{S}_n)_x$. We claim that if $\dim \mathcal{R}_n(x) = \dim \mathcal{R}_{n+D(n)}(x)$ and $\dim \mathcal{L}_n(x) = \dim \mathcal{L}_{n+D(n)}(x)$, then the sequences $\mathcal{R}_i(x), \mathcal{L}_i(x)$, and $\mathcal{S}_i(x)$ stabilize at $n + D(n)$. Indeed, for any $i \in [n, n + D(n) - 1]$, if $\dim \mathcal{L}_i(x) = \dim \mathcal{L}_{i+1}(x)$, then $\mathcal{L}_i(x) = \mathcal{L}_{i+1}(x)$ and similarly for $\mathcal{R}_i(x)$. Since $(\mathcal{S}_{i+1})_x$ is a subgroup of $(\mathcal{S}_i)_x$ and they have the same Lie algebra, either $\mathcal{S}_i(x) = \mathcal{S}_{i+1}(x)$ or $\mathcal{S}_{i+1}(x)$ has fewer connected components than $\mathcal{S}_i(x)$. It follows that there is $i \in [n, n + D(n) - 1]$ such that $\mathcal{S}_i(x) = \mathcal{S}_{i+1}(x)$. It now follows that the sequences of functions $\mathcal{R}_n, \mathcal{L}_n$, and \mathcal{S}_n stabilize for n big enough. Once \mathcal{R}_n and \mathcal{S}_n stabilize, we can keep f_n and h_n unchanged.

It remains to construct f_n and h_n . We start with $n = 0$. Fix a finite extension L of K and a prime \mathfrak{q} of L , that satisfies the conditions of Lemma 3.8. Denote the set $\mathcal{X}(L_{\mathfrak{q}})$ by X . For every character θ of the first congruence Lie algebra $\mathfrak{g}^1(O_{L,\mathfrak{q}})$, let $X_{\theta} = \{x \in X \mid \Pi_0(x) = \theta\}$, and let X_{nt} be the union of all X_{θ} for all non-trivial characters θ . Denoting the orbit method map by Ω and using Lemma 3.10, we have

$$\begin{aligned} \zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) &= \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) + \sum_{\theta \neq 1} \frac{1}{|\theta^{\mathbf{G}(O_{L,\mathfrak{q}})}|} \dim(\Omega(\theta))^{-s} \zeta_{\mathbf{G}(O_{L,\mathfrak{q}})|\Omega(\theta)}(s) = \\ &= \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) + \sum_{\theta \neq 1} \int_{X_{\theta}} \frac{1}{\lambda(X_{\theta})} \frac{1}{|\Pi_{\{0\}}(x)^{\mathbf{G}(O_{L,\mathfrak{q}})}|} \dim(\Xi_{\{0\}}(x))^{-s} \zeta_{\mathbf{G}(O_{L,\mathfrak{q}})|\Xi_{\{0\}}(x)}(s) d\lambda(x) = \\ &= \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) + \int_{X_{nt}} \frac{1}{\lambda(\Pi_{\{0\}}^{-1}(\Pi_{\{0\}}(x)))} \frac{1}{|\Pi_{\{0\}}(x)^{\mathbf{G}(O_{L,\mathfrak{q}})}|} \dim(\Xi_{\{0\}}(x))^{-s} \zeta_{\mathbf{G}(O_{L,\mathfrak{q}})|\Xi_{\{0\}}(x)}(s) d\lambda(x). \end{aligned}$$

By Theorem 4.3, Corollary 4.12, and Proposition 4.14, there are quantifier-free definable functions $\varphi_1, \varphi_2, \varphi_3 : \mathcal{X} \rightarrow \Gamma$ such that

1. $\dim(\Xi_{\{0\}}(x)) = \left| \Pi_{\{0\}}(x)^{\mathbf{G}^{(1)}(O_{L,\mathfrak{q}})} \right|^{1/2} = |O_L/\mathfrak{q}|^{\varphi_1(x)}.$
2. $\dim(\mathcal{L}_1(x)) = |O_L/\mathfrak{q}|^{\varphi_2(x)}.$
3. $\lambda(\Pi_{\{0\}}^{-1}(\Pi_{\{0\}}(x))) = |O_L/\mathfrak{q}|^{\varphi_3(x)}.$

By Lemma 3.15 there is a constant C such that, for every x ,

$$|\Pi_{\{0\}}(x)^{\mathbf{G}(O_{L,\mathfrak{q}})}| = |\Pi_{\{0\}}(x)^{\mathbf{G}^{(1)}(O_{L,\mathfrak{q}})}| \cdot |\mathbf{G}(k)/(\mathcal{S}_1)_x(L_{\mathfrak{q}})| \sim_C |O_L/\mathfrak{q}|^{2\varphi_1(x) + \dim \mathfrak{g} - \varphi_2(x)}.$$

Therefore,

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - 1 \sim_C \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) - 1 + \int_{X_{nt}} |O_L/\mathfrak{q}|^{-\varphi_3(x) - (2\varphi_1(x) + \dim \mathfrak{g} - \varphi_2(x)) - \varphi_1(x)s} \zeta_{\mathbf{G}(O_{L,\mathfrak{q}})|\Xi_{\{0\}}(x)} d\lambda(x),$$

and we can take $f_0 = -\varphi_3(x) - (2\varphi_1(x) + \dim \mathfrak{g} - \varphi_2(x))$, $g_0 = \varphi_1(x)$.

Finally, we construct f_{n+1}, h_{n+1} from f_n, h_n . Fix L , a prime \mathfrak{q} of O_L different from any of the finitely many primes omitted, a nilpotent Lie algebra $A \subset \mathfrak{g}(O_L/\mathfrak{q})$, and a character θ of \tilde{A} (recall that \tilde{A} is the pre-image of A under the map $\mathfrak{g}(O_{L,\mathfrak{q}}) \rightarrow \mathfrak{g}(O_L/\mathfrak{q})$; it is a Lie ring, and, in particular, an additive group).

Consider the set

$$X_{A,\theta} = \Pi_{\mathcal{R}_n}^{-1}(\theta) = \{x \in \mathcal{X}(L_{\mathfrak{q}}) | \mathcal{R}_n(x) = A, \Pi_{\mathcal{R}_n}(x) = \theta\}.$$

On $X_{A,\theta}$, the values of $(\mathcal{S}_n)_x(L_{\mathfrak{q}})$, $(\mathcal{S}_{n+1})_x(L_{\mathfrak{q}})$, and $\mathcal{R}_{n+1}(x)$ are constant. Denote them by S_n, S_{n+1} , and B respectively. Let τ_1, \dots, τ_m be the characters of \tilde{B} that extend θ , and let

$$X_{A,\theta}^i = \Pi_{\mathcal{R}_{n+1}}^{-1}(\tau_i) = \{x \in X_{A,\theta} | \Pi_B(x) = \tau_i\}.$$

The $X_{A,\theta}^i$ form a partition of $X_{A,\theta}$ into $m = [B : A]$ parts with equal measure. From Lemma 3.10 and Clifford theory we get that, for every $x \in X_{A,\theta}$,

$$\zeta_{(\tilde{S}_n)_x(L_{\mathfrak{q}})|\Xi_{\mathcal{R}_n}(x)}(s) = \zeta_{\tilde{S}_n|\Omega(\theta)}(s) = [\tilde{S}_n : \widetilde{S_{n+1}}]^{-s} \sum_{i=1}^m \frac{|\theta^{\exp \widetilde{\mathcal{R}_{n+1}}}|}{|\tau_i^{\widetilde{S_{n+1}}}|} \left(\frac{\dim \Omega(\tau_i)}{\dim \Omega(\theta)} \right)^{-s} \zeta_{\widetilde{S_{n+1}}|\Omega(\tau_i)}(s).$$

Therefore,

$$\begin{aligned} \int_{X_{A,\theta}} \zeta_{\tilde{S}_n(x)|\Xi_{\mathcal{R}_n}(x)}(s) d\lambda(x) &= [S_n : S_{n+1}]^{-s} \sum_{i=1}^m \int_{X_{A,\theta}^i} \frac{\lambda(X_{A,\theta})}{\lambda(X_{A,\theta}^i)} \cdot \frac{|\theta^{\exp \widetilde{\mathcal{R}_{n+1}}(x)}|}{|\tau_i^{\widetilde{S_{n+1}}}|} \left(\frac{\dim \Omega(\tau_i)}{\dim \Omega(\theta)} \right)^{-s} \zeta_{\widetilde{S_{n+1}}|\Omega(\tau_i)}(s) d\lambda(x) \\ &= \sum_i^m \int_{X_{A,\theta}^i} [(\mathcal{S}_n)_x(L_{\mathfrak{q}}) : (\mathcal{S}_{n+1})_x(L_{\mathfrak{q}})]^{-s} \cdot [\mathcal{R}_{n+1}(x) : \mathcal{R}_n(x)] \cdot \frac{|\Pi_{\mathcal{R}_n}(x)^{\exp \widetilde{\mathcal{R}_{n+1}}(x)}|}{|\Pi_{\mathcal{R}_{n+1}}(x)^{(\widetilde{S_{n+1}})_x(L_{\mathfrak{q}})}|} d\lambda(x). \end{aligned}$$

$$\cdot \left(\frac{\dim \Xi_{\mathcal{R}_{n+1}}(x)}{\dim \Xi_{\mathcal{R}_n}(x)} \right)^{-s} \zeta_{(\widetilde{\mathcal{S}_{n+1}})_x(L_q) | \Xi_{\mathcal{R}_{n+1}}(x)}(s) d\lambda(x).$$

There are quantifier-free definable functions $\psi_1, \psi_2, \psi_3, \psi_4 : \mathcal{X} \rightarrow \Gamma$ and a constant C , all independent of L and \mathfrak{q} , such that, for all $x \in \mathcal{X}(L_q)$,

1. $[(\mathcal{S}_n)_x(L_q) : (\mathcal{S}_{n+1})_x(L_q)] \sim_C |O_L/\mathfrak{q}|^{\psi_1(x)}$, (by Corollary 4.12 and Lemma 3.15).
2. $[\mathcal{R}_{n+1}(x) : \mathcal{R}_n(x)] = |O_L/\mathfrak{q}|^{\psi_2(x)}$, (because \mathcal{R}_n and \mathcal{R}_{n+1} are quantifier-free definable).
3. $\frac{|\Pi_{\mathcal{R}_n}(x)^{\exp \widetilde{\mathcal{R}_{n+1}}(x)}|}{|\Pi_{\mathcal{R}_{n+1}}(x)^{(\widetilde{\mathcal{S}_{n+1}})_x(L_q)}|} \sim_C |O_L/\mathfrak{q}|^{\psi_3(x)}$
(because $|\Pi_{\mathcal{R}_{n+1}}(x)^{(\widetilde{\mathcal{S}_{n+1}})_x(L_q)}| = |\Pi_{\mathcal{R}_{n+1}}(x)^{\exp \widetilde{\mathcal{R}_{n+1}}(x)}| \cdot [(\mathcal{S}_n)_x(L_q) : (\mathcal{S}_{n+1})_x(L_q)]$, and by Proposition 4.13).
4. $\frac{\dim \Xi_{\mathcal{R}_{n+1}}(x)}{\dim \Xi_{\mathcal{R}_n}(x)} \sim_C |O_L/\mathfrak{q}|^{\psi_4(x)}$
(because, for example, $\dim \Xi_{\mathcal{R}_n}(x) = |\Pi_{\mathcal{R}_n}(x)^{\exp \widetilde{\mathcal{R}_n}(x)}|^{1/2}$ and by Proposition 4.13).

Denote $\alpha_n = \psi_2 + \psi_3$ and $\beta_n = \psi_1 + \psi_4$. We have that

$$\int_{X_{A,\theta}} \zeta_{(\widetilde{\mathcal{S}_n})_x(L_q) | \Xi_{\mathcal{R}_n}(x)}(s) d\lambda(x) \sim_{C^3} \int_{X_{A,\theta}} |O_L/\mathfrak{q}|^{\alpha_n(x) + \beta_n(x)s} \zeta_{(\widetilde{\mathcal{S}_{n+1}})_x(L_q) | \Xi_{\mathcal{R}_{n+1}}(x)}(s) d\lambda(x)$$

Defining $f_{n+1} = f_n + \alpha_n$ and $h_{n+1} = h_n + \beta_n$,

$$\begin{aligned} \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) - 1 &\sim_{C_n} \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) - 1 + \int_{\mathcal{X}(L_q)} |O_L/\mathfrak{q}|^{f_n(x) + h_n(x)s} \cdot \zeta_{(\widetilde{\mathcal{S}_n})_x(L_q) | \Xi_{\mathcal{R}_n}(x)} d\lambda(x) = \\ &= \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) - 1 + \sum_{A,\theta} \int_{X_{A,\theta}} |O_L/\mathfrak{q}|^{f_n(x) + h_n(x)s} \cdot \zeta_{(\widetilde{\mathcal{S}_n})_x(L_q) | \Xi_{\mathcal{R}_n}(x)} d\lambda(x) \sim_{C^3} \\ &\sim_{C^3} \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) - 1 + \sum_{A,\theta} \int_{X_{A,\theta}} |O_L/\mathfrak{q}|^{f_{n+1}(x) + h_{n+1}(x)s} \cdot \zeta_{(\widetilde{\mathcal{S}_{n+1}})_x(L_q) | \Xi_{\mathcal{R}_{n+1}}(x)} d\lambda(x) = \\ &= \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) - 1 + \int_{\mathcal{X}(L_q)} |O_L/\mathfrak{q}|^{f_{n+1}(x) + h_{n+1}(x)s} \cdot \zeta_{(\widetilde{\mathcal{S}_{n+1}})_x(L_q) | \Xi_{\mathcal{R}_{n+1}}(x)} d\lambda(x), \end{aligned}$$

and the result follows with $C_{n+1} = C_n C^3$. \square

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $\mathcal{S}_n, \mathcal{R}_n, \mathcal{L}_n, h_n, f_n, C_n$ be the sequences constructed in Theorem 4.15. Suppose N is large enough so that the $\mathcal{S}_n, \mathcal{R}_n, \mathcal{L}_n, h_n, f_n$ stabilize for $n \geq N$.

$$\zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) - 1 \sim_{C_N} \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) - 1 + \int_{\mathcal{X}(L_{\mathfrak{q}})} |O_L/\mathfrak{q}|^{f_N(x)+h_N(x)s} \cdot \zeta_{(\widetilde{\mathcal{S}_N})_x(L_{\mathfrak{q}})|\Xi_{\mathcal{R}_N}(x)}(s) d\lambda(x). \quad (5)$$

The groups $\mathbf{G}_{O_L/\mathfrak{q}}$ are simply connected, for all but finitely many primes \mathfrak{q} . Theorem 2.7 implies that $\zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) - 1 \sim_C \xi_{a,|O_L/\mathfrak{q}|}(s)$ for some $a \in \mathcal{A}$ and $C \in \mathbb{R}$. Since there are definable functions $\psi_1, \psi_2 : \mathcal{O}^{\dim \mathbf{G}+1} \rightarrow \Gamma$ such that, for every L and \mathfrak{q} ,

$$\xi_{a,|O_L/\mathfrak{q}|}(s) = \int_{\mathcal{O}^{\dim \mathbf{G}+1}(L_{\mathfrak{q}})} |O_L/\mathfrak{q}|^{\psi_1(x)+\psi_2(x) \cdot s} d\lambda(x),$$

it is enough to show that there are definable functions $\psi_3, \psi_4 : \mathcal{X} \rightarrow \Gamma$ such that, for every L and almost all \mathfrak{q} , the integrand in (5) is approximated by $|O_L/\mathfrak{q}|^{\psi_3(x)+\psi_4(x)}$.

For every $x \in \mathcal{X}(L_{\mathfrak{q}})$, $(\mathcal{S}_N)_x$ is an algebraic group which is the stabilizer of $\Xi_{\mathcal{R}_N}(x)$, that $\mathcal{L}_N(x)$ is the Lie algebra of $(\mathcal{S}_N)_x$, and that $\mathcal{R}_N(x)$ is the nilpotent radical of $\mathcal{L}_N(x)$. Let $(\mathcal{S}_N)_x^0$ denote the connected component of the identity. It was shown in Proposition 4.6 that there is a constant D_1 such that $[(\mathcal{S}_N)_x : (\mathcal{S}_N)_x^0] \leq D_1$ for all $x \in \mathcal{X}$. We get that

$$\zeta_{(\widetilde{\mathcal{S}_N})_x(L_{\mathfrak{q}})|\Xi_{\mathcal{R}_N}(x)} \sim_{D_1} \zeta_{(\widetilde{\mathcal{S}_N})_x^0(L_{\mathfrak{q}})|\Xi_{\mathcal{R}_N}(x)}.$$

The group $(\mathcal{S}_N)_x^0 / \exp \mathcal{R}_N(x)$ is a reductive group of bounded dimension over the field O_L/\mathfrak{q} , and so, by Lemma 3.6, there are only finitely many primes for which the Schur multiplier of any of the groups $((\mathcal{S}_N)_x^0 / \exp \mathcal{R}_N(x))(O_L/\mathfrak{q})$, for $x \in \mathcal{X}(L_{\mathfrak{q}})$, contains an element of order $\text{char}(O_L/\mathfrak{q})$. Hence, by Lemmas 3.4 and 3.3,

$$\zeta_{(\widetilde{\mathcal{S}_N})_x^0(L_{\mathfrak{q}})|\Xi_{\mathcal{R}_N}(x)}(s) = \zeta_{(\mathcal{S}_N)_x^0(L_{\mathfrak{q}})/\exp \mathcal{R}_N(x)}(s).$$

Finally, by Proposition 3.13, there is a quantifier-free partition of \mathcal{X} such that, on each part, the root system of $(\mathcal{S}_N)_x^0 / \exp \mathcal{R}_N(x)$ is constant. In each such part, by Theorem 2.7, there is $a \in \mathcal{A}$ and a constant D_2 such that $\zeta_{(\mathcal{S}_N)_x^0(L_{\mathfrak{q}})/\exp \mathcal{R}_N(x)} \sim_{D_2} 1 + \xi_{a,|O_L/\mathfrak{q}|}$. We get the claim with $C = D_1 D_2 C_N$. \square

5 Quantifier-free Integrals

Let K be a number field. We consider the first-order language of valued fields together with a constant (of the valued field sort) for every element of K .

Theorem 5.1. *Let $X \subset \mathcal{O}^n$ be a quantifier-free definable set, and let $f, g : X \rightarrow \Gamma$ be quantifier-free definable functions. Then there are quasi-affine varieties W_i $i = 1, \dots, N$ defined over K , and integers A_{ij}, B_{ij} $i = 1, \dots, N, j = 1, \dots, n_i$ such that, for any finite field extension $K \subset L$ and for almost all primes \mathfrak{q} of O_L , we have*

$$\int_{X(L_{\mathfrak{q}})} |O_L/\mathfrak{q}|^{sf(x)+g(x)} d\lambda(x) = \sum_{i=1}^N |W_i(O_L/\mathfrak{q})| \cdot \prod_{j=1}^{n_i} \frac{|O_L/\mathfrak{q}|^{A_{ij}s+B_{ij}}}{1 - |O_L/\mathfrak{q}|^{A_{ij}s+B_{ij}}}, \quad (6)$$

where λ is the normalized Haar measure on $O_{L,\mathfrak{q}}^n$.

Proof. The set X is a disjoint union of finitely many sets defined by formulas of the form

$$(H(x) = 0) \wedge \phi(\text{ac}(H'(x))) \wedge \psi(\text{val}(H''(x))),$$

where H, H', H'' are polynomial functions with coefficients in K , ϕ is a quantifier-free formula in the language of fields, and ψ is a quantifier-free formula in the language of ordered groups. It is enough to prove the theorem assuming X is of this form. If $H(x)$ is nonzero, then the integral on the left hand side is zero, for all \mathfrak{q} , so the result follows. In the following, we assume this is not the case. The formula ϕ is equivalent to the conjunction of equations and inequalities that define a quasi-affine algebraic set. Similarly, the formula ψ is equivalent to a Boolean combination of affine inequalities defining a disjoint union of rational polyhedral cones. Hence, we can assume that X is defined by a formula of the form

$$((\text{ac}(H'(x))) \in Y) \wedge ((\text{val}(H''(x))) \in Q), \quad (7)$$

where Y is a quasi-affine set, and Q is a rational cone.

We now turn to the integrand on the left hand side of (6). It follows from elimination of quantifiers in the theory of divisible ordered abelian groups ([27, Corollary 3.1.17]) that, for every definable function $w : \Gamma^m \rightarrow \Gamma$, there is a partition of Γ^m into rational polyhedral cones so that w is linear on each cone. After a further subdivision of X , if necessary, we may assume that the functions f, g are defined by formulas of the form

$$f(x) = \sum_i r_i \text{val}(F_i(x)) \quad ; \quad g(x) = \sum_i r'_i \text{val}(G_i(x)),$$

where $r_i, r'_i \in \mathbb{Q}$ and $F_i(x), G_i(x)$ are polynomials with coefficients in K .

Suppose that $H'(x) = (H'_1(x), \dots, H'_M(x))$ and $H''(x) = (H''_1(x), \dots, H''_M(x))$, where the H'_i 's and H''_i 's are polynomials. Let Υ be the product of all the polynomials H'_i, H''_i, F_i, G_i . By Hironaka's theorem on resolution of singularities, applied to the ideal generated by Υ , there is a finite set T of primes of O_K , a scheme \mathfrak{X} , defined over the

localization $(O_K)_T$, and a projective map $\pi = (\pi_1, \dots, \pi_n) : \mathfrak{X}_K \rightarrow \mathbb{A}_K^n$ such that \mathfrak{X}_K is smooth and, if V denotes the pre-image of the zero locus of Υ by π , then V is a divisor with normal crossings. Let $\{X_r\}$ be the set of all Boolean combinations of the components of V in \mathfrak{X}_K . The collection $\{X_r\}$ is a partition of \mathfrak{X}_K into quasi-projective varieties, and, for each r , there is a coordinate system x_1, \dots, x_n of \mathfrak{X}_K in a neighborhood U_r of X_r such that, in U_r , the following condition holds:

Condition 5.2. *Each of the functions $\text{Jac}_\pi = \det\left(\frac{\partial \pi_i}{\partial x_j}\right), H'_i \circ \pi, H''_i \circ \pi, F_i \circ \pi, G_i \circ \pi$ is the product of an invertible function and a monomial in the functions x_1, \dots, x_n .*

By quasi-compactness of X , there are a finite set S of primes of O_K , which contains T , and $(O_K)_S$ -models of U_r, X_r , and x_j (which we will continue to denote by the same letter) such that the U_r, X_r are smooth over $\text{Spec}(O_K)_S$, the x_j s are local coordinate systems on U_r , the X_r cover $\mathfrak{X} \times \text{Spec}(O_K)_S$, and Condition 5.2 above still holds (see [11, Theorem 2.4] for a similar claim).

Suppose that, on U_r , we have $\text{Jac}_\pi(z) = \alpha(z) \cdot \prod x_i(z)^{n_i}$, where α is invertible on U_r . By Condition 5.2, α is invertible modulo \mathfrak{p} , for every prime \mathfrak{p} not in S . In particular, if the reduction of z modulo \mathfrak{p} belongs to $U_r(O/\mathfrak{p})$, then the valuation of $\alpha(z)$ is zero. Similarly, we can assume this holds for all the invertible functions in Condition 5.2.

Suppose that L is a finite extension of K . Let \mathfrak{q} be a prime of L that lies over a prime \mathfrak{p} of K . Assume that $\mathfrak{p} \notin S$, and that the extension $L_{\mathfrak{q}}/K_{\mathfrak{p}}$ is unramified. Denote the reduction map from $\mathfrak{X}(O_{L,\mathfrak{q}})$ to $\mathfrak{X}(O_L/\mathfrak{q})$ by red . The set $\mathfrak{X}(O_{L,\mathfrak{q}})$ is partitioned into the sets $\text{red}^{-1}(X_r(O_L/\mathfrak{q}))$. For each r , the coordinate system x_1, \dots, x_n gives a local homeomorphism of $\text{red}^{-1}(X_r(O_L/\mathfrak{q}))$ into an open set in $O_{L,\mathfrak{q}}^n$. If we denote the pull-back of the normalized Haar measure of $O_{L,\mathfrak{q}}^n$ along this local homeomorphism by μ , we get that the Radon–Nikodym derivative of $\pi^*\lambda$ with respect to μ is equal to $|O_L/\mathfrak{q}|^{\text{val}(\text{Jac}_\pi(x))}$. Hence,

$$\begin{aligned} \int_{X(L_{\mathfrak{q}})} |O_L/\mathfrak{q}|^{f(x)s+g(x)} d\lambda(x) &= \int_{\mathfrak{X}((O_L)_{\mathfrak{q}})} 1_{X(L_{\mathfrak{q}})}(\pi(x)) \cdot |O_L/\mathfrak{q}|^{f \circ \pi(x)s+g \circ \pi(x)} d\pi^*\lambda(x) = \\ &= \sum_r \int_{\text{red}^{-1}(X_r(O_L/\mathfrak{q}))} 1_{X(L_{\mathfrak{q}})}(\pi(x)) \cdot |O_L/\mathfrak{q}|^{f \circ \pi(x)s+g \circ \pi(x)+\text{val}(\text{Jac}_\pi(x))} d\mu(x) \end{aligned} \quad (8)$$

On each of the sets $\text{red}^{-1}(X_r(O_L/\mathfrak{q}))$, Condition 5.2 above implies that there are integers $a_j, b_j, c_j, d_{i,j}, e_{i,j}$ and maps η_i, θ_i on U_r such that, for all $z \in \text{red}^{-1}(U_r(O_L/\mathfrak{q}))$,

$$1. \quad f \circ \pi(z) = \sum_{j=1}^n a_j \text{val}(x_j(z)).$$

2. $g \circ \pi(z) = \sum_{j=1}^n b_j \text{val}(x_j(z)).$
3. $\text{val}(\text{Jac}_\pi(z)) = \sum_{j=1}^n c_j \text{val}(x_j(z)).$
4. $H'_i \circ \pi(z) = \eta_i(z) \cdot \prod_{j=1}^n x_j(z)^{d_{i,j}}.$
5. $H''_i \circ \pi(z) = \theta_i(z) \cdot \prod_{j=1}^n x_j(z)^{e_{i,j}}.$
6. $\text{val}(\eta_i(z)) = \text{val}(\theta_i(z)) = 0.$

Fix r and restrict to $z \in \text{red}^{-1}(X_r(O_L/\mathfrak{q}))$. Combining 4., 5., 6., and Formula (7), we get that the condition $\pi(z) \in X(L_{\mathfrak{q}})$ is equivalent to the conjunction of the condition $\left(\text{ac}(\eta_i(z)) \prod_j \text{ac}(x_j(z)^{d_{i,j}})\right) \in Y$ and the condition $\left(\sum_j e_{i,j} \text{val}(x_j(z))\right) \in Q$.

For each i , the function $\eta_i(z)$ is a rational function in the coordinates $x_j(z)$. Therefore, the function $\text{ac}(\eta_i(z)) = \text{red}(\eta_i(z))$ is a rational function in the functions $\text{red}(x_j(z))$. If we decompose $\text{red}^{-1}(X_r(O_L/\mathfrak{q}))$ according to whether $\text{red}(x_j(z))$ is zero or non-zero, we get a partition of $\mathfrak{X}(O_{L,\mathfrak{q}})$, such that, on each piece, $\text{ac}(\eta_i(z))$ depends only on $\text{ac}(x_j(z))$. It follows that there are quasi-affine schemes W_t defined over $(O_K)_S$ such that, on the piece of the partition of $\mathfrak{X}(O_{L,\mathfrak{q}})$ indexed by t , the condition $\left(\text{ac}(\eta_i(z)) \prod_j \text{ac}(x_j(z)^{d_{i,j}})\right) \in Y$ is equivalent to $(\text{ac}(x_j(z))) \in W_r(O_L/\mathfrak{q})$.

Denote the linear map given by the matrix $(e_{i,j})$ by E . The sum in (8) is equal to

$$\begin{aligned} \sum_r \sum_{(\alpha_j) \in W_r(O_L/\mathfrak{q})} \sum_{(\gamma_j) \in E^{-1}(Q)} |O_L/\mathfrak{q}|^{-\sum_j \gamma_j + \sum_j b_j \gamma_j + \sum_j c_j \gamma_j + s \sum_j a_j \gamma_j} = \\ \sum_r |W_r(O_L/\mathfrak{q})| \sum_{(\gamma_j) \in E^{-1}(Q)} |O_L/\mathfrak{q}|^{\sum_j (b_j + c_j - 1) \gamma_j + s \sum_j a_j \gamma_j}, \end{aligned}$$

which is of the form we want, as rational polyhedral cones can be decomposed into simple cones (see [3, Theorem 6.2] and [10, Theorem 11.1.9]). \square

Proof of Theorem 2.8. By Theorem 4.1, there is a quantifier-free definable set \mathscr{Y} , quantifier-free definable functions $f_1, f_2 : \mathscr{Y} \rightarrow \Gamma$, and a constant C , such that, for every finite extension $K \subset L$ and almost every prime \mathfrak{q} of O_L ,

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - 1 \sim_C \int_{\mathscr{Y}(L_{\mathfrak{q}})} |O_L/\mathfrak{q}|^{f_1(x) + f_2(x) \cdot s} d\lambda(x).$$

By Theorem 5.1, we get that, for almost every \mathfrak{q} ,

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - 1 \sim_C \sum_i |W_i(O_L/\mathfrak{q})| \cdot \prod_j \frac{|O_L/\mathfrak{q}|^{A_{ij}s + B_{ij}}}{1 - |O_L/\mathfrak{q}|^{A_{ij}s + B_{ij}}},$$

where the W_i are quasi-affine varieties defined over K . We can assume that the W_i are irreducible. By the Lang–Weil estimates, there is a constant D such that, for almost all primes \mathfrak{q} , either $W_i(O_L/\mathfrak{q})$ is empty, or

$$\frac{1}{2} \leq \frac{|W_i(O_L/\mathfrak{q})|}{|O_L/\mathfrak{q}|^{\dim W_i}} \leq D.$$

Moreover, the Cebotarev Density Theorem implies that the set of primes \mathfrak{q} for which $W_i(O_L/\mathfrak{q})$ is non-empty has positive analytic density. Denoting

$$\beta = \max_{i,j} \left\{ -\frac{B_{ij}}{A_{ij}} \right\},$$

and

$$b_i = \left(\dim(W_i) + \sum_j B_{ij}, \sum_j A_{ij} \right) \in \mathcal{A},$$

we get that the abscissa of convergence of $\zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s)$ is less than or equal to β and, for $\operatorname{Re}(s) \geq \beta$, the two claims of Theorem 2.8 hold. \square

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